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Vector braids

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Abstract

In this paper we define a new family of groups which generalize the *classical braid groups on* $\mathbb{C}$. We denote this family by $\{B_n^m\}_{n\geq m+1}$ where $n,m\in\mathbb{N}$. The family $\{B_n^1\}_{n\in\mathbb{N}}$ is the set of classical braid groups on n strings. The group B_n^m is related to the set of motions of n unordered points in $\mathbb{C}^m$, so that at any time during the motion, each $m+1$ of the points span the whole of $\mathbb{C}^m$ in the sense of affine geometry. There is a map from B_n^m to the symmetric group on n letters. We let P_n^m denote the kernel of this map. In this paper we are mainly interested in finding a presentation of and understanding the group P_n^2. We give a presentation of a group PL_n which maps surjectively onto P_n^2. We also show the surjection $PL_n \rightarrow P_n^2$ induces an isomorphism on first and second integral homology and conjecture that it is an isomorphism. We then find an infinitesimal presentation of the group P_n^2. Finally, we also consider the analagous groups where points lie in $\mathbb{P}^m$ instead of $\mathbb{C}^m$. These groups generalize of the classical braid groups on the sphere. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\mathbb{A}^m$ denote m-dimensional, complex affine space. Let X_n^m be the space of ordered n-tuples of elements of $\mathbb{A}^m$, with $n \geq m+1$ such that each $m+1$ of the components of each n-tuple span the whole of $\mathbb{A}^m$ in the sense of affine geometry. The symmetric group on n letters, Σ_n, acts on X_n^m via permuting the components of each point. This action is fixed point free, and so we can form the quotient space X_n^m/Σ_n.

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Definition 1.1. Let $B_n^m = \pi_1(X_n^m/\Sigma_n)$ and $P_n^m = \pi_1(X_n^m)$. Call these groups the group of n stringed vector braids on $\mathbb{A}^m$ and the group of n-stringed pure vector braids on $\mathbb{A}^m$, respectively.

The long exact sequence of a fibration gives us the short exact sequence

$$1 \to P_n^m \to B_n^m \to \Sigma_n \to 1. \tag{1}$$

The space X_n^1 is the well-known *configuration space* of n points in $\mathbb{C}$ [7]. In general, we can describe the space X_n^m as

$$X_n^m = \underbrace{\mathbb{A}^m \times \cdots \times \mathbb{A}^m}_{n} - \Delta,$$

where

$$\Delta = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{A}^m, \text{ span } \{x_{i_1}, \dots, x_{i_{m+1}}\} \neq \mathbb{A}^m\}.$$

In the case $m = 1$, the set Δ is simply the "fat diagonal". In general, we call Δ the *determinental variety*. Choose coordinates $(x_1, \dots, x_n)$ for $\mathbb{A}^m$, i.e. an isomorphism between $\mathbb{A}^m$ and $\mathbb{C}^m$. Let $x_i = (z_{1i}, \dots, z_{mi})$, where $1 \leq i \leq n$ and $z_{ij} \in \mathbb{C}$. Then the defining equations of Δ are all possible $(m+1) \times (m+1)$ minors

$$\Delta_{i_1 \dots i_{m+1}} = \begin{vmatrix} 1 & \cdots & 1 \\ z_{1i_1} & \cdots & z_{1i_{m+1}} \\ \vdots & & \vdots \\ z_{mi_1} & \cdots & z_{mi_{m+1}} \end{vmatrix}$$

of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}.$$

Remark 1.2. Later, we also consider motions of n points in $\mathbb{P}^m$ instead of $\mathbb{A}^m$ (see Section 2).

The group P_n^1 is the classical *pure braid group of n strings on* $\mathbb{C}$. The pure braid group has a very nice presentation, which may be understood geometrically [1]. The main aim of this paper is to discover a geometrical presentation for the group P_n^2 analogous to that of the classical pure braid group. Elements of P_n^2 may be thought of as motions of n points in $\mathbb{A}^2$ so that at any time during this motion no three of the n points lie on a line.

We now informally state the main results of this paper. We define a group PL_n via a presentation, and a surjective homomorphism $\varphi_n : PL_n \to P_n^2$ (note that, in this context, PL_n does *not* denote piecewise linear homeomorphisms of real Euclidean n-space). The presentation of PL_n is given in Definition 7.3. Theorems 7.14 and 7.20 state that the homomorphism φ_n induces isomorphisms on the first and second integral homology groups. In light of the fact that both PL_n and P_n^2 are finitely generated, that both groups have a presentation with the number of generators equal to the rank of the first homology of the group with integer coefficients, and that the map $PL_n \to P_n^2$ is surjective, the following conjecture [1] seems reasonable.

Conjecture 1.3. *The homomorphism* $\varphi_n : PL_n \to P_n^2$ *is an isomorphism.*

Despite the fact that it appears to be difficult to prove Conjecture 1.3, it is relatively straightforward to find an infinitesimal presentation for P_n^2. In Proposition 8.1 we prove that the completion of the group algebra $\mathbb{C}[P_n^2]$ with respect to the augmentation ideal is isomorphic to the non-commutative power series ring in the indeterminates X_{ijk}, $1 \leq i < j < k \leq n$, modulo the two sided ideal generated by the relations

$$[X_{ijk}, X_{rst}] \quad \text{when } i,j,k,r,s,t \text{ are distinct,}$$

$$[X_{ijk}, X_{rsk}] \quad \text{when } i,j,k,r,s \text{ are distinct,}$$

$$[X_{ijk}, X_{jkl} + X_{ikl} + X_{ijl}] \quad \text{when } i,j,k,l \text{ are distinct}$$

and

$$[X_{rst}, X_{1ij} + \cdots + X_{i-1,ij} + X_{i,i+1,j} + \cdots + X_{i,j-1,j} + X_{ij,j+1} + \cdots + X_{ijn}],$$

which holds when rst is one of the triples

$$\{1ij\}, \ldots, \{i-1, ij\}, \{i, i+1, j\}, \ldots, \{i, j-1, j\}, \{ij, j+1\}, \ldots, \{ijn\},$$

where $1 < i < j < n$.

We now summarize the contents of this paper. First, it is helpful to recall one method for finding a presentation of the classical pure braid group, $P_n = P_n^1$. Recall that the map

$$p_n^1 : X_n^1 \to X_{n-1}^1,$$

which forgets the last point in each n-tuple is a fibration with fiber equal to $\mathbb{C}$ less $n-1$ points [7]. For $n \geq 3$, the long exact sequence for a fibration provides us with the following sequence:

$$1 \to L_{n-1} \to P_n \to P_{n-1} \to 1, \tag{2}$$

where L_{n-1} is a free group on $n-1$ generators. In [6] it is shown that this sequence is split. Let a_{in}, $1 \leq i \leq n-1$ denote the loop in which the nth point goes around the ith point in the punctured $\mathbb{C}$. The set of loops $\{a_{in} \mid 1 \leq i \leq n-1\}$ generates L_{n-1}. We picture the loop a_{in} as an element of P_n in Fig. 1.

[1] It appears that Tomohide Terasoma from Tokyo Metropolitan University has recently proven this conjecture to be true [22].

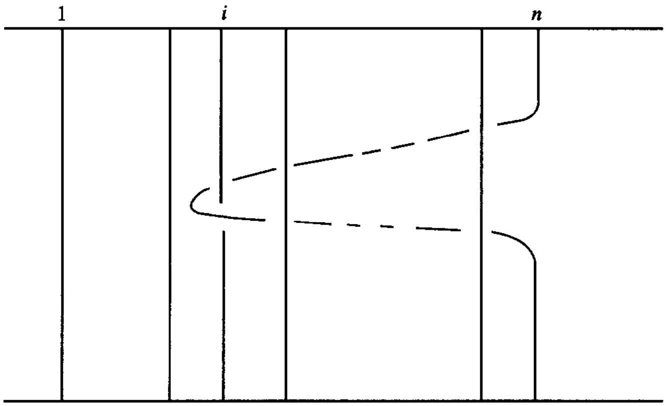

Fig. 1. The generator a_{in} of P_n.

Using sequence (2) and the fact that $P_2 = \mathbb{Z}$, we can inductively prove that the group P_n admits a presentation with generators

$$a_{ij}, \quad 1 \leq i < j \leq n$$

and defining relations,

$$a_{ij}^{-1}\, a_{rs}\, a_{ij} = \begin{cases} a_{rs} & \text{if } r<i<j<s \text{ or } i<j<r<s, \\ a_{is}\, a_{rs}\, a_{is}^{-1} & \text{if } i<r=j<s, \\ a_{is}\, a_{js}\, a_{rs}\, a_{js}^{-1}\, a_{is}^{-1} & \text{if } r=i<j<s, \\ a_{is}\, a_{js}\, a_{is}^{-1}\, a_{js}^{-1} a_{rs}\, a_{js}\, a_{is}\, a_{js}^{-1}\, a_{is}^{-1} & \text{if } i<r<j<s. \end{cases}$$

Since the generators of P_{n-1} clearly lift to P_n, finding a presentation for P_n at each stage involves three main operations. First, we have to add the generators from the fiber group L_{n-1} to the group P_{n-1}. Second, we have to add relations to P_n obtained by conjugating each generator of L_{n-1} by the generators of P_{n-1}. Finally, we have to lift relations from P_{n-1} to P_n. Note that this last step is particularly simple since, as we mentioned above, sequence (2) is split.

The way in which we find presentations for the groups PL_n will be modelled on this approach, although, as we shall see, there will be significant complications.

In Section 2 we define the anologues of P_n^m with $\mathbb{A}^m$ replaced by $\mathbb{P}^m$, and compare these groups to P_n^m and B_n^m, respectively. In Section 3 we discuss the groups P_n^m and B_n^m for $m \leq n+2$. In Section 4 we show that the map

$$p_n^2 : X_n^2 \to X_{n-1}^2$$

defined by forgetting the last point in each n-tuple is not a fibration for $n \geq 5$. However, we will be able to repair this defect in some sense by using the fact that the map p_n^2

is a fibration except over a set of complex codimension one. A consequence of this fact is Corollary 4.2, which states that there is an exact sequence

$$\pi_1(GF_n^2) \to P_n^2 \to P_{n-1}^2 \to 1, \tag{3}$$

where GF_n^2 denotes the generic fiber of the map p_n^2.

We use this sequence to find relations within the group P_n^2. However, there are three major complications which did not occur when we found a presentation for the classical pure braid group P_n. First, the fiber is more intricate than in the braid case: it is the complement of lines in $\mathbb{A}^2$ rather than a punctured copy of $\mathbb{C}$. Second, and more importantly, we do not know if the group $\pi_1(GF_n^2)$ injects into P_n^2. If this were the case, then Conjecture 1.3 would be true. Third, we have not been able to show that sequence (3) is split, in contrast to the sequence (2) (in fact, it appears unlikely that the map p_n^2 has a section). Thus, it is necessary to develop a new technique, called the *reciprocity law*, for lifting relations from P_{n-1}^2 to P_n^2.

In Section 5 we give a way of describing loops in the complexification of a real arrangement of lines in $\mathbb{C}^2$. In Section 6 we use some techniques from stratified Morse theory to find nice presentations for the fiber groups, $\pi_1(GF_n^2)$, and we also analyze the relationship between $\pi_1(GF_n^2)$ and $\pi_1(F_k)$, where F_k denotes the generic fiber of the projection $p_k^2 : X_n^2 \to X_{n-1}^2$, $1 \le k \le n$, which is defined by forgetting the kth point in each n-tuple. In Section 7 we define the group PL_n and a surjective homomorphism $\varphi_n : PL_n \to P_n^2$. We then state the main theorems of this paper, Theorems 7.2, 7.14 and 7.20, and prove Theorems 7.14 and 7.20. In Section 8 we find an infinitesimal presentation for the group P_n^2. In Section 9 we describe the consequences of considering motions of points in $\mathbb{P}^2$ as opposed to $\mathbb{A}^2$.

The remaining sections of this paper are devoted to the proof of Theorem 7.2. In Section 10 we see how to conjugate the generators of $\pi_1(GF_n^2)$ by the generators of P_{n-1}^2. In Section 11 we describe a move within the group P_n^2, called the *reciprocity law*, which we use to lift relations from P_{n-1}^2 to P_n^2. This law is justified in Section 13. Finally, in Section 12 we explain how to lift relations from P_{n-1}^2 into P_n^2.

2. Affine versus projective

Our definition of X_n^m involved looking at points in $\mathbb{A}^m$. By thinking of $\mathbb{A}^m$ as being the affine part of $\mathbb{P}^m$, we may extend our definitions to motions of points in $\mathbb{P}^m$.

Let us first consider the classical braid groups. We denote the classical pure braid group of n strings on $\mathbb{P}^1$ by Q_n. Since $\mathbb{P}^1$ is homeomorphic to the two sphere we also see that Q_n is the classical pure braid group on the sphere. By considering $\mathbb{C}$ to be the affine part of $\mathbb{P}^1$ we get a surjective map $P_n \to Q_n$. Note that the following relations, which do not hold in P_n, hold in Q_n:

$$a_{12}\, a_{13} \ldots a_{1n} = 1,$$

$$a_{1k}\, a_{2k}\, \ldots\, a_{k-1,k} a_{k,k+1} \ldots a_{kn} = 1 \quad \text{for } 2 \le k \le n. \tag{4}$$

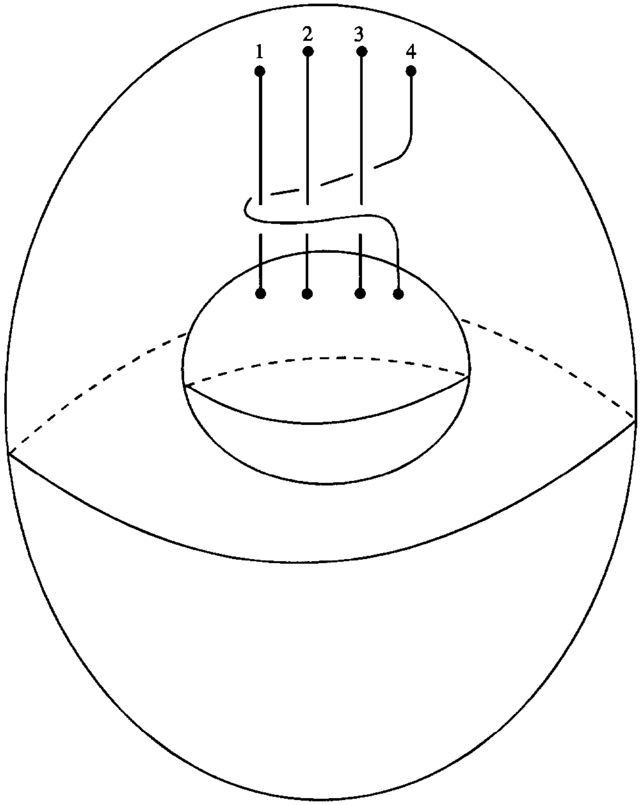

Fig. 2. The product $a_{12}a_{13}a_{14}$ is trivial in Q_4.

(for example see Fig. 2). In fact, if we let Y_n^1 denote the configuration space of n distinct points in $\mathbb{P}^1$, then, using the long exact sequence of the fibration $q_n^1 : Y_n^1 \to Y_{n-1}^1$ obtained by forgetting the nth point [7], it can be proven that these are all of the extra relations that need to be added to the presentation of P_n in order to obtain a presentation of Q_n (these extra relations arise since the fiber of q_n^1 is homeomorphic to a sphere less $n-1$ points, as opposed to $\mathbb{A}^1$ less $n-1$ points). It is also interesting to note that the introduction of these relations into the presentation of P_n introduces torsion (e.g. the center of Q_n contains an element of order 2 – see Corollary 2.6).

We now generalize these notions to our situation. Let Y_n^m denote the space of ordered n-tuples in $\mathbb{P}^m$, with $n \geq m+1$ so that each $m+1$ of the points of each n-tuple span the whole of $\mathbb{P}^m$. As with X_n^m, the symmetric group on n letters acts fixed point freely on Y_n^m by permuting the components of each point. Define $C_n^m = \pi_1(Y_n^m/\Sigma_n)$ and $Q_n^m = \pi_1(Y_n^m)$. Call these groups *the group of n-stringed vector braids on* $\mathbb{P}^m$ and *the pure group of n-stringed vector braids on* $\mathbb{P}^m$, respectively. Note that we have natural maps $P_n^m \to Q_n^m$ and $B_n^m \to C_n^m$.

Lemma 2.1. *The natural map* $P_n^m \to Q_n^m$ *is surjective.*

Proof. This follows as X_n^m is a Zariski open subset of the smooth variety Y_n^m. □

Corollary 2.2. *The natural map* $B_n^m \to C_n^m$ *is surjective.*

We now define an action of the affine and projective linear groups on X_n^m and Y_n^m. The affine group[2] $AGL_{m+1}(\mathbb{C})$ acts on the space X_n^m via the diagonal action. If $n \geq m+1$, the isotropy group of a point is trivial. If $n = m+1$ then $AGL_{m+1}(\mathbb{C})$ acts transitively. It follows that X_{m+1}^m is diffeomorphic to $AGL_{m+1}(\mathbb{C})$. Let $\overline{X_n^m}$ denote the quotient space $X_n^m/AGL_{m+1}(\mathbb{C})$. Then $AGL_{m+1}(\mathbb{C}) \to X_n^m \to \overline{X_n^m}$ is a principal $AGL_{m+1}(\mathbb{C})$ bundle. Moreover, it has a section (cf. [11, p. 421]). Hence, we have the following result.

Lemma 2.3. *The space* X_n^m *is diffeomorphic to* $\overline{X_n^m} \times AGL_{m+1}(\mathbb{C})$.

Corollary 2.4. *The group* P_n^m *has a central element of infinite order.*

Proof. We have $\pi_1(X_n^m) \cong \pi_1(\overline{X_n^m}) \times \pi_1(AGL_{m+1}(\mathbb{C}))$. But $\pi_1(AGL_{m+1}(\mathbb{C}))$ is isomorphic to $\mathbb{Z}$. □

The group $PGL_{m+1}(\mathbb{C})$ acts on the space Y_n^m via the diagonal action. If $n \geq m+2$ the isotropy group of a point is trivial. If $n = m+2$ then $PGL_{m+1}(\mathbb{C})$ acts transitively. It follows that Y_{m+2}^m is diffeomorphic to $PGL_{m+1}(\mathbb{C})$. Let $\overline{Y_n^m}$ denote the quotient space $Y_n^m/PGL_{m+1}(\mathbb{C})$. Then $PGL_{m+1}(\mathbb{C}) \to Y_n^m \to \overline{Y_n^m}$ is a principal $PGL_{m+1}(\mathbb{C})$ bundle. It has a section (cf. [11, p. 421]). Hence, we have the following result.

Lemma 2.5. *The space* Y_n^m *is diffeomorphic to* $\overline{Y_n^m} \times PGL_{m+1}(\mathbb{C})$.

Corollary 2.6. *The group* Q_n^m *has a central element of order* $m+1$.

Proof. We have $\pi_1(Y_n^m) \cong \pi_1(\overline{Y_n^m}) \times \pi_1(PGL_{m+1}(\mathbb{C}))$. But $\pi_1(PGL_{m+1}(\mathbb{C}))$ is isomorphic to $\mathbb{Z}/(m+1)\mathbb{Z}$. □

We shall see later that the $AGL_{m+1}(\mathbb{C})$ action on X_n^m and the $PGL_{m+1}(\mathbb{C})$ action on Y_n^m are useful in understanding some of the properties of the groups P_n^m and Q_n^m.

3. Getting started

In this section we study the groups P_n^m, Q_n^m, B_n^m and C_n^m when $n \leq m+2$.

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[2] The affine group $AGL_{m+1}(\mathbb{C})$ is defined to be the stabilizer of the line at infinity in $PGL_{m+1}(\mathbb{C})$. Hence, $AGL_{m+1}(\mathbb{C})$ is the semidirect product of $GL_m(\mathbb{C})$ by $\mathbb{C}^m$.

Proposition 3.1. *The groups P_m^m and Q_{m+1}^m are trivial for all $m \in \mathbb{N}$.*

Proof. We begin by extending the definition of the space X_n^m to the case where $n \leq m$. Let X_n^m, $1 \leq n \leq m-1$, be the space of n-tuples of points in $\mathbb{A}^m$ such that the m-tuple spans an affine subspace of maximal dimension. The map $X_n^m \to X_{n-1}^m$ obtained by forgetting the last point in each n-tuple is a fibration. The fiber of this map is equal to $\mathbb{A}^m$ less an affine subspace of complex codimension $m-(n-1)$. Hence, the fiber of each of these maps has trivial fundamental group when $n<m$. Since X_1^m is diffeomorphic to $\mathbb{A}^m$ for each m, we can use the exact sequence of a fibration to inductively show that, for fixed m, X_m^m has trivial homotopy groups when $n<m$.

A similar argument applies to the space Y_n^m, giving us the result for Q_{m+1}^m. □

Corollary 3.2. *The natural homomorphisms*

$$B_m^m \to \Sigma_m \quad \text{and} \quad C_{m+1}^m \to \Sigma_{m+1}$$

are isomorphisms.

The underlying reason for why the group Q_{m+1}^m is trivial and P_{m+1}^m is not is that the space $\mathbb{A}^m$ less a hyperplane is homotopy equivalent to S^1, whereas the space $\mathbb{P}^m$ less a hyperplane is contractible.

Proposition 3.3. *The group P_{m+1}^m is isomorphic to $\mathbb{Z}$ for all $m \in \mathbb{N}$.*

Proof. The map $X_{m+1}^m \to X_m^m$ is a fibration, with fiber equal to $\mathbb{C}^2$ less a line. This space is a $K(\mathbb{Z},1)$. Since $\pi_i(X_m^m)$, is trivial for $i \geq 1$ (Proposition 3.1), we obtain the result using the long exact sequence of a fibration. □

We immediately get a similar result for Q_{m+2}^m as a consequence of Lemma 2.5.

Proposition 3.4. *The group P_{m+2}^m is isomorphic to $\mathbb{Z}/(m+1)\mathbb{Z}$, for all $m \in \mathbb{N}$.*

We now use the $PGL_{m+1}(\mathbb{C})$ action on the space Y_n^m to find a presentation of the group C_{m+2}^m.

Proposition 3.5. *The group C_{m+2}^m admits a presentation with generators*

$$\sigma_1, \ldots, \sigma_{m+1}, \tau$$

and defining relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j|>1,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq m,$$

$$\sigma_i \tau = \tau \sigma_i, \quad 1 \leq i \leq m+1,$$

$$\sigma_i^2 = \tau, \quad 1 \le i \le m+1,$$
$$\tau^{m+1} = 1.$$

Proof. In Section 2 we saw that the spaces Y^m_{m+2} and $PGL_{m+1}(\mathbb{C})$ are diffeomorphic. Let e_i, $1 \le i \le m+1$ be the standard basis of $\mathbb{C}^{m+1}$, and let $g \in PGL_{m+1}(\mathbb{C})$. Then the map

$$\theta : PGL_{m+1}(\mathbb{C}) \to Y^m_{m+2},$$
$$\theta : g \mapsto (ge_1, \dots, ge_{m+1}, ge_1 + \cdots + ge_{m+1})$$

is a diffeomorphism. We know that Σ_{m+2} acts on the space Y^m_{m+2} on the right by permuting coordinates. We now see that Σ_{m+2} also acts on the right of $PGL_{m+1}(\mathbb{C})$. We do this by embedding Σ_{m+2} into $PGL_{m+1}(\mathbb{C})$.

Let s_i, $1 \le i \le m+1$ denote the transposition $(i, i+1)$. Then Σ_{m+2} has presentation with generators $s_1, \dots, s_{m+1}$, and relations

$$s_i s_j = s_j s_i, \quad |i-j| > 1,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad 1 \le i \le m,$$
$$s_i^2 = 1, \quad 1 \le i \le m+1.$$

Let $P_i \in PGL_{m+1}(\mathbb{C})$ denote the coset of the permutation matrix corresponding to the transposition s_i (i.e. the identity matrix with its ith and $(i+1)$th columns swapped). Map the element s_i to P_i for $1 \le i \le m$. Let $P_{m+1} \in PGL_{m+1}(\mathbb{C})$ be the coset of the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}.$$

Map the element s_{m+1} to P_{m+1}. Matrix computations show that this map may be extended to an injective homomorphism from Σ_{m+2} to $PGL_{m+1}(\mathbb{C})$.

Let Σ_{m+2} act on the right of $PGL_{m+2}(\mathbb{C})$ by group multiplication. Note that

$$\begin{aligned}\theta(gP_i) &= (gP_ie_1, \dots, gP_ie_{m+1}, gP_i(e_1 + \cdots + e_{m+1})) \\ &= (ge_{s_i(1)}, \dots, eq_{s_i(m+1)}, ge_{s_i(1)} + \cdots + ge_{s_i(m+1)}) \\ &= \theta(g)s_i,\end{aligned}$$

when $1 \le i \le m$. This is because P_ie_j is equal to the jth column of P_i. Also,

$$\theta(gP_{m+1}) = (e_1, \dots, e_m, -g(e_1 + \cdots + e_{m+1}), -ge_{m+1}).$$

Thus, θ is Σ_{m+2} equivariant and $PGL_{m+2}(\mathbb{C})/\Sigma_{m+2}$ is diffeomorphic to Y^m_{m+2}/Σ_{m+2}.

Note that $PGL_{m+2}(\mathbb{C})$ is isomorphic to $PSL_{m+1}(\mathbb{C})$. Consequently, $SL_n(\mathbb{C})$ is the universal cover of $PGL_n(\mathbb{C})$. The natural homomorphism $SL_{m+1}(\mathbb{C}) \xrightarrow{\pi} PSL_{m+1}(\mathbb{C})$ is a $\mathbb{Z}/(m+1)\mathbb{Z}$ covering whose kernel is generated by the diagonal matrix τ whose diagonal entries are all equal to $\exp(2\pi i/(m+1))$. Let $G=\pi^{-1}(\Sigma_{m+2})$, i.e. the pull-back of the natural extension $SL_{m+1}(\mathbb{C})\to PSL_{m+1}(\mathbb{C})$ along the embedding that we chose for Σ_{m+1} into $PGL_{m+1}(\mathbb{C})$. Then we have the short exact sequence

$$1\to \mathbb{Z}/(m+1)\mathbb{Z} \xrightarrow{\pi} G\to \Sigma_{m+2}\to 1.$$

We use this now to show that G is given by the same presentation as that stated in the theorem. Let $\omega=\exp(2\pi i/2(m+1))$ and $\sigma_i:=\omega P_i$, $1\le i\le m+1$. Then each of these matrices lies in $SL_{m+1}(\mathbb{C})$. Use the matrix σ_i as a lift of P_i for $1\le i\le m+1$ and the matrix τ as the generator of the image of π. Simple matrix calculations show that the stated relations between the matrices σ_i, $1\le i\le m+1$, and τ hold in G.

To complete the proof we have to show that G is isomorphic to C^m_{m+2}. First, note that since $\mathbb{Z}/(m+1)\mathbb{Z}$ is central in G we have the isomorphism

$$SL_{m+1}(\mathbb{C})/G \cong [\mathbb{Z}/(m+1)\mathbb{Z} \setminus SL_{m+1}(\mathbb{C})]/\Sigma_{m+2}.$$

But $[\mathbb{Z}/(m+1)\mathbb{Z}]\setminus SL_{m+1}(\mathbb{C})$ is isomorphic to $PSL_{m+1}(\mathbb{C})$, which is in turn isomorphic to Y^m_{m+2}. Hence, $SL_{m+1}(\mathbb{C})/G$ is isomorphic to Y^m_{m+2}/Σ_{m+2}. Since $SL_{m+1}(\mathbb{C})$ is the universal cover of $SL_{m+1}(\mathbb{C})/G$, we conclude that

$$G\cong \pi_1(Y^m_{m+2}/\Sigma_{m+2})\cong C^m_{m+2}. \qquad \square$$

Remark 3.6. Note that the group C^m_{m+2} of Proposition 3.5 has a central cyclic subgroup of order $m+1$, with cokernel given by the symmetric group on $m+1$ letters. Such extensions are characterized by the second cohomology group of the symmetric group with coefficients in $\mathbb{Z}/(m+1)\mathbb{Z}$. Thus, if m is even, the extension is trivial, so that C^m_{m+2} is isomorphic to a direct product of Σ_{m+2} and $\mathbb{Z}/(m+1)\mathbb{Z}$.

4. Forgetting a point

Let $p^m_n: X^m_n\to X^m_{n-1}$ denote the map which takes n-tuples in X^m_n to $(n-1)$-tuples in X^m_{n-1} by forgetting the nth point. In [7] it is shown that the map p^1_n is a fibration for all $n\ge 2$. However, in general, these maps fail to be fibrations when m is greater than one. For example, in the case $m=2$ we have the following result.

Proposition 4.1. *The map p^2_n is not a fibration for $n\ge 5$.*

Proof. First, consider the case when $n=5$. Let $(x_1,\dots,x_n)\in X^2_n$ and let L_{ij}, $1\le i<j\le n$, denote the line through the points x_i and x_j, in the fiber of p^2_{n+1} over $(x_1,\dots,x_n)$. Then the fiber over the point $(x_1,\dots,x_n)$ will be equal to $\mathbb{A}^2$ less the union of the lines L_{ij}. The homotopy type of the fibers will not be constant since some of the fibers will

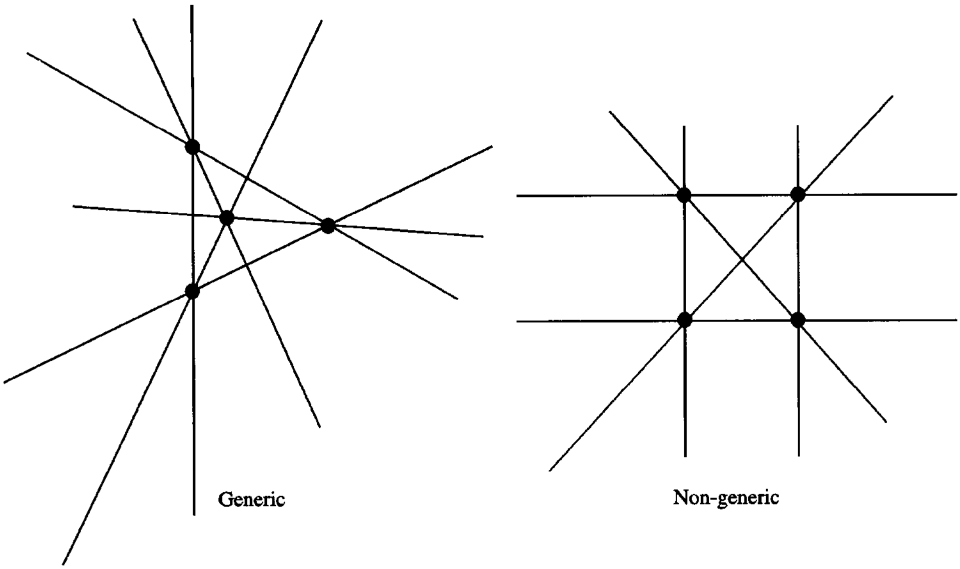


Fig. 3. Parallel problems.

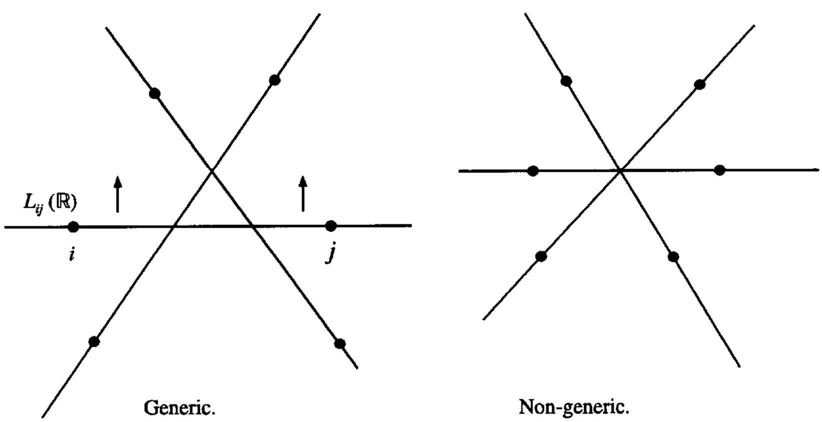


Fig. 4. How degenerate fibers can occur.

contain parallel lines, which do not intersect in $\mathbb{A}^2$ (see Fig. 3 for a non-generic fiber of the map $p_5^2 : X_5^2 \to X_4^2$). This problem occurs for all $n \geq 5$.

In the case when $n \geq 7$ another type of degeneration also occurs. We refer to Fig. 4. This is a real picture of the fibers of p_n^2 which shows how one may get non-generic fibers. When lines with disjoint indices intersect only in double points we are in the generic situation. However, we see in Fig. 4 that as the line L_{ij} moves upwards it passes through a double point giving us a triple intersection. The homotopy type of the fiber changes when we obtain triple points (for example, even the fundamental group of the fiber changes [16]); we no longer have a fibration. □

Given that the map p_n^2 fails to be a fibration when $n \geq 5$, it might seem hopeless to use the same method that we used in Section 1 to find a presentation of the pure braid group in finding a presentation for the group P_n^2. However, we are able to partially salvage this situation using results contained in [14]. Since $p_n^m : X_n^m \to X_{n-1}^m$ is a surjective algebraic map, it is a topological fibration except over a subset of X_{n-1}^m of complex codimension one [14, Comment 0.4, p. 95], which we call the *discriminant locus*. The fibers of the map p_n^m degenerate over this locus as described in the previous proposition. Moreover, if we choose a basepoint in the complement of this subset and let GF_n^m denote the fiber over this point (i.e. the generic fiber), then [14, Lemma, p. 103] implies the following result.

Lemma 4.2. *The sequence*

$$\pi_1(GF_n^m) \to \pi_1(X_n^m) \xrightarrow{(p_n^m)_*} \pi(X_{n-1}^m) \to 1$$

is exact.

In the case $m=2$ we have the exact sequence

$$\pi_1(GF_n^2) \to P_n^2 \to P_{n-1}^2 \to 1 \tag{5}$$

for each $n \geq 4$. This sequence is analogous to the short exact sequence (2), where $n \geq 3$, involving the classical pure braid groups. Note that if the group $\pi_1(GF_n^2)$ injects into P_n^2, then Conjecture 1.3 is true.

We close this section by noting that we also have the map $q_n^m : Y_n^m \to Y_{n-1}^m$, obtained by forgetting the nth point. Proposition 4.1 is also true for the map q_n^m. In fact, we can say slightly more in this case.

Proposition 4.3. *The map $q_n^2 : Y_n^2 \to Y_{n-1}^2$ is a fibration for $n=4,5$ and 6. However, the map q_n^2 is not a fibration for $n \geq 7$.*

Proof. First, consider the cases when n is equal to 4,5 and 6. Let $(x_1,\dots,x_n) \in X_n^2$ and let L_{ij}, $1 \leq i<j \leq n$, denote the line through the points x_i and x_j, in the fiber of p_{n+1}^2 over $(x_1,\dots,x_n)$. Then the fiber over the point $(x_1,\dots,x_n)$ will be equal to $\mathbb{P}^2$ less the union of the lines L_{ij}. When $n=5$ and 6, the lines L_{ij} and L_{kl} only intersect in a double point when $\{i,j\} \cap \{k,l\} = \emptyset$. Since the combinatorics of all of the fibers are the same, they are all diffeomorphic by [12, Theorem 4.3]. As we are fibering over a connected manifold, the proof follows.

In the case when $n \geq 7$ we can use the same argument to the one that given in Proposition 4.1 using Fig. 4. □

When $n \leq 6$, define F_n^2 to be the fiber of the map $q_n^2 : Y_n^2 \to Y_{n-1}^2$ (in view of Proposition 4.3, this makes sense). Then we have the following lemma.

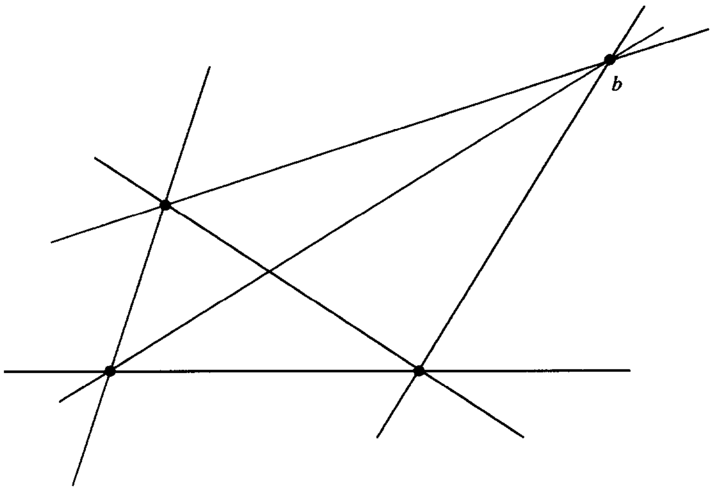


Fig. 5.

Lemma 4.4. *For $n=5$ and 6 the sequence*

$$1 \to \pi_1(F_n^2) \to Q_n^2 \to Q_{n-1}^2 \to 1.$$

is exact.

Proof. Since Y_4^2 is diffeomorphic to $PGL_{m+1}(\mathbb{C})$ the group $\pi_2(Y_4^2)$ is trivial. Thus, the case $n=5$ is an immediate consequence of Proposition 4.1 and the long exact sequence of fibration.

The case $n=5$ also yields the exact sequence

$$\pi_2(F_5^2) \to \pi_2(Y_5^2) \to \pi_2(Y_4^2).$$

We now show that F_5^2 is a $K(\pi_1(F_5^2),1)$ space, which implies that the group $\pi_2(X_5^2)$ vanishes. Consider the pencil of lines in F_5^2, through the point b (see Fig. 5). This pencil fibers F_5^2 (i.e. if we define a map from F_5^2 to b by sending each line in the pencil to b, then this map is a fibration). The base is $\mathbb{P}^1$ less three points and the fiber is $\mathbb{P}^1$ less four points. Hence, F_5^2 is a $K(\pi,1)$ space.

By applying Proposition 4.1 and using the long exact sequence of a fibration once more we obtain the result for the case $n=6$. $\square$

5. Getting around

In this section we find generators for the fundamental group of the complement of a set of complexified real lines in $\mathbb{A}^2$. Some of the material in this section is drawn from [18].

We begin by stating some conventions that we will use from now on. Fix a real structure on $\mathbb{A}^n$. We denote the real points of this structure by $\mathbb{A}^n(\mathbb{R})$. If V is an affine linear subspace of $\mathbb{A}^n$ then let $V(\mathbb{R})$ be equal to $V \cap \mathbb{A}^n(\mathbb{R})$.

Let $\{L_i(\mathbb{R})\}$ be a set of *oriented* lines in $\mathbb{A}^2(\mathbb{R})$. Denote the union of the $L_i(\mathbb{R})$ by $\mathscr{A}(\mathbb{R})$. Let (x_1,x_2) be coordinates for $\mathbb{A}^2(\mathbb{R})$. Then $(x_1+\mathrm{i}y_1, x_2+\mathrm{i}y_2)$, $y_1,y_2 \in \mathbb{R}$, are coordinates for $\mathbb{A}^2$. Once we have chosen coordinates for $\mathbb{A}^2$ we will abuse notation and write $\mathbb{C}^2$ for $\mathbb{A}^2$ and $\mathbb{R}^2$ for $\mathbb{A}^2(\mathbb{R})$. Let $\varepsilon > 0$ be an arbitrary real number. It will be convenient to work in the tubular neighborhood $N_\varepsilon = \{(x_1+\mathrm{i}y_1, x_2+\mathrm{i}y_2) \in \mathbb{C}^2 \mid y_1^2+y_2^2 \leq \varepsilon\} - \mathscr{A}$ of $\mathbb{R}^2 - \mathscr{A}(\mathbb{R})$. To justify this we require the following proposition.

Proposition 5.1. *For all $\varepsilon > 0$, the set N_ε is homotopy equivalent to $\mathbb{C}^2 - \mathscr{A}$.*

Proof. We prove this fact using stratified Morse theory [8]. Begin by stratifying the space $\mathbb{C}^2$ using the arrangement $\mathscr{A}$ (for details see [8, p. 245]). Let $f : \mathbb{C}^2 \to \mathbb{R}$ be the function defined by the formula $f(x_1+\mathrm{i}y_1, x_2+\mathrm{i}y_2) = y_1^2 + y_2^2$. This is a Morse function (in the sense of stratified Morse theory) on the stratified space $\mathbb{C}^2$. Let $X_{>t}$ denote the set of points x in $\mathbb{C}^2 - \mathscr{A}$ such that $f(x) > t$. Outside the set of points $x \in \mathbb{C}^2$ where $f(x) < \varepsilon$ the function f has no critical points. Hence, the set $X_{>t}$ is homotopy equivalent to the set N_ε for any $t \geq \varepsilon$. □

Let $L(\mathbb{R})$ be a generic, oriented line in $\mathbb{R}^2 - \mathscr{A}(\mathbb{R})$ and let v be a vector which orients $L(\mathbb{R})$. Its complexification has a canonical orientation and the orientation of the frame formed by the vectors v and $\mathrm{i}v$ agrees with this orientation. Let a be any point in $L(\mathbb{R})$. Then we let $\tilde{a} \in L$ denote the point which is distance ε from a in the direction $\mathrm{i}v$. Let $p \in L(\mathbb{R}) - \mathscr{A}(\mathbb{R})$ and $q \in L(\mathbb{R}) \cap \mathscr{A}(\mathbb{R})$. We now define a loop in N_ε based at the point p (see Fig. 6). First, move in the direction $\mathrm{i}v$ from point p to point $\tilde{p}$. Then move along the real line in L, which joins $\tilde{p}$ and $\tilde{q}$ towards $\tilde{q}$. On reaching the point $\tilde{q}$, pick a loop l_q with center q and of radius ε within L. Now, go around this loop in the positive direction with respect to the orientation of L. Finally, return to p along the same path taken to $\tilde{q}$ from the point p. We call this loop *the loop in L, based at p, which goes around the point q.*

Remark 5.2. The homotopy class of the loop which we have just defined depends only upon the choices of $L(\mathbb{R})$, its orientation and the points p and q.

Denote the set of points in $L(\mathbb{R}) \cap \mathscr{A}(\mathbb{R})$ by $\{q_i\}$. Let ζ_i be the loop in L, based at p which goes around q_i. Theorems of Lefschetz and Zariski (cf. [8, Ch. 2]) immediately imply the following result.

Proposition 5.3. *The set of loops $\{\zeta_i\}$ generates $\pi_1(\mathbb{C}^2 - \mathscr{A}, p)$.*

Remark 5.4. If we instead considered $\mathscr{A}(\mathbb{R})$ as being an arrangement of lines in $\mathbb{P}^2(\mathbb{R})$, then the loops ζ_i would generate $\pi_1(\mathbb{P}^2 - \mathscr{A}, p)$ (cf. Lemma 2.1).

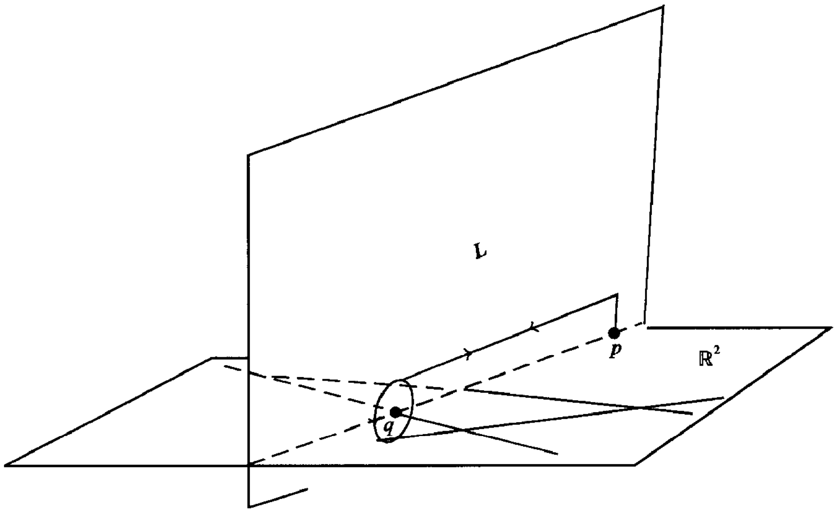


Fig. 6. The loop in L, based at p, which goes around the point q.

We now need a pair of lemmas which will help us manipulate loops in the complement of a set of complexified real lines in $\mathbb{A}^2(\mathbb{R})$. To do this we first define a *hop*. Let $H(\mathbb{R})$ an arbitrary real line in $\mathbb{A}^2(\mathbb{R})$. Pick a point $p \in \mathbb{A}^2(\mathbb{R}) - H(\mathbb{R})$ near to the line $H(\mathbb{R})$. The line $H(\mathbb{R})$ divides $\mathbb{A}^2(\mathbb{R})$ into two regions. Let $q \in \mathbb{A}^2(\mathbb{R})$ be a point in $\mathbb{A}^2(\mathbb{R}) - H(\mathbb{R})$ close to p that lies in the connected component of $\mathbb{A}^2(\mathbb{R}) - H(\mathbb{R})$ not containing the point p. To hop from the point p to the point q in $\mathbb{A}^2 - H$, choose a line $L(\mathbb{R})$ joining the points p and q and follow a loop in L from p to q, in the negative direction with respect to the canonical orientation of L.

The first lemma is local in nature. Let p be an element of $\mathbb{A}^2(\mathbb{R})$. Suppose that n lines $L_j(\mathbb{R})$, $1 \le j \le n$, pass through the point p. Label the lines from 1–n in anticlockwise order. Choose a line $L(\mathbb{R})$ passing through p, which lies between the lines $L_1(\mathbb{R})$ and $L_n(\mathbb{R})$. Let C be a small circle in $\mathbb{A}^2(\mathbb{R})$ centered at p. Let $a, b \in L(\mathbb{R})$ denote the two points of intersection of the circle C with $L(\mathbb{R})$. (see Fig. 7). Let γ be the path in $\mathbb{A}^2 - \bigcup L_j$, joining the points a and b, which is obtained by following the circle C in the anticlockwise direction and hopping over each line $L_j(\mathbb{R})$. Let γ' be the path in $\mathbb{A}^2 - \bigcup L_j$, joining points a and b, which is obtained by following the circle C in the clockwise direction and hopping over each line $L_j(\mathbb{R})$.

Lemma 5.5. *The paths γ and γ' are homotopic in $\mathbb{A}^2 - \bigcup L_j$, relative to their endpoints a and b.*

Proof. Let v be the vector in $L(\mathbb{R})$ from the point a to the point p. Let P_v be the plane $\mathbb{A}^2(\mathbb{R}) + iv$. Then the intersection $P_v \cap L_j$ is empty for $1 \le j \le n$. Hence, the intersection $P_v \cap [\bigcup L_j]$ is empty.

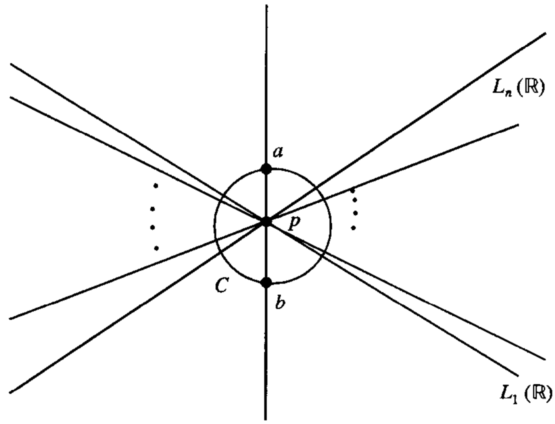


Fig. 7.

If p is an element of $\mathbb{A}^2(\mathbb{R})$ then let $\tilde{p}$ denote the point $a + \mathrm{i}v$ in P_v. Let $C + \mathrm{i}v$ denote the circle in P_v which lies above C. Let ρ be the path obtained by going in direction $\mathrm{i}v$ from a to $\tilde{a}$, going along $C + \mathrm{i}v$ in the anticlockwise direction and finally by going in direction $-\mathrm{i}v$ to b. The path ρ is homotopic to γ relative to the points a and b. Let ρ' be the path obtained by going in direction $\mathrm{i}v$ from a to $\tilde{a}$, going along $C + \mathrm{i}v$ in the clockwise direction and finally by going in direction $-\mathrm{i}v$ to b. The path ρ' is homotopic to γ' relative to the points a and b.

To conclude the proof we see that the paths ρ and ρ' are homotopic. This is because the paths contained in $C + \mathrm{i}v$ which were used to define ρ and ρ' are homotopic in P_v relative to $\tilde{a}$ and $\tilde{b}$. □

The second lemma is global in nature. Let D be a bounded region contained in $\mathbb{A}^2(\mathbb{R})$ which is diffeomorphic to a real closed disc in $\mathbb{R}^2$. Also, assume that the boundary of D is smooth. Let $\mathscr{A}(\mathbb{R})$ be an arrangement of real lines contained in $\mathbb{A}^2(\mathbb{R})$ such that each line in $\mathscr{A}(\mathbb{R})$ is tranverse to the boundary of D and the number of components of $D - \mathscr{A}(\mathbb{R})$ is finite. Also, assume that no three lines in $\mathscr{A}(\mathbb{R})$ intersect in a point in D and that the boundary of D contains none of the multiple points of $\mathscr{A}(\mathbb{R})$ (see Fig. 8) (thus, all of the intersection points of the arrangement $\mathscr{A}(\mathbb{R})$ contained within D are *double points*). Let $M(\mathbb{R})$ denote the set $D - \mathscr{A}(\mathbb{R})$ and let M be the set

$$\{u + \mathrm{i}v \mid u \in M(\mathbb{R}), v \in \mathbb{R}^2, \text{ and } \|v\| < \varepsilon\}.$$

We now define some loops in M. Let $\{L_\alpha(\mathbb{R}) \mid \alpha \in A\}$, denote the set of line segments in $M(\mathbb{R})$, obtained by intersecting $\mathscr{A}(\mathbb{R})$ with the set D. Let p be a base point of $M(\mathbb{R})$. For each line segment $L_\alpha(\mathbb{R})$ we define a loop l_α in M, based at p, as follows. Pick a point q on the line segment $L_\alpha(\mathbb{R})$ which lies in between any two intersection points. Let $l_\alpha(\mathbb{R})$ be a path in D joining the point p and q which intersects each line segment $L_\alpha(\mathbb{R})$ transversely only once and avoids all intersection points. We now define the

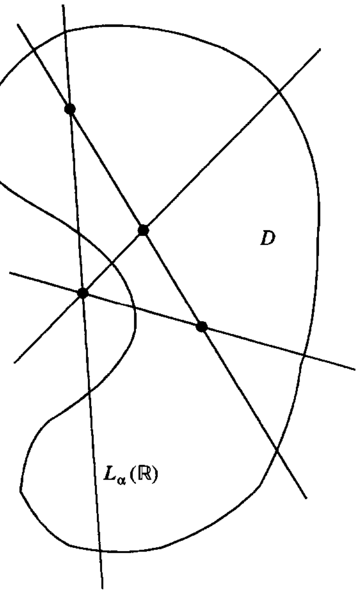


Fig. 8.

loop l_α. Follow the path $l_\alpha(\mathbb{R})$, hopping over any line segment, until reaching the line $L_\alpha(\mathbb{R})$. Then choose a line $L(\mathbb{R})$ passing through q, which is transverse to $L_\alpha(\mathbb{R})$. Now follow a small loop in L, which encircles L_α, in the positive direction with respect to the orientation of L. Finally, return to the point p along the same path which was taken outward from p.

Lemma 5.6. *The group $\pi_1(M,p)$ admits a presentation with generators l_α, $\alpha \in A$ and relations*

$$[l_\alpha, l_\beta] = 1 \quad \textit{if } L_\alpha(\mathbb{R}) \cap L_\beta(\mathbb{R}) \neq \emptyset.$$

Proof. First, we show that there exists a Morse function on the set M, in the sense of stratified Morse theory [8]. To do this we take a family of real expanding sets $B_t(\mathbb{R})$, $0 \le t \le 1$, such that $B_0(\mathbb{R}) = \{p\}$ and $B_1(\mathbb{R}) = M(\mathbb{R})$, which grow in the following way. The family first intersects each line segment $L_\alpha(\mathbb{R})$ tangentially, and it also envelops each double point one at a time. By taking the distance from p to points in the set B_t we construct the required Morse function on M.

Now, using stratified Morse theory, we find a presentation for the group $\pi_1(M,p)$. First, we have to add a generator l'_α for each line segment L_α when the family $B_t(\mathbb{R})$ first crosses line segment L_α. The generator l'_α can be chosen to be the composition of a path, which hops over each line segment once, from p to $L_\alpha(\mathbb{R})$ and a loop which goes around $L_\alpha(\mathbb{R})$, by following a small loop in a complex line L, in the positive direction with respect to the orientation of L. Also, using Morse theory again, on encountering each double point $L_\alpha(\mathbb{R}) \cap L_\beta(\mathbb{R}) \neq \emptyset$, we have to add a commutator relation $[l'_\alpha, l'_\beta] = 1$. Note that these are the only relations that we require, as there are no other intersection points of $\mathscr{A}(\mathbb{R})$ in $M(\mathbb{R})$.

To complete the proof we show that the loops l'_α and l_α are homotopic in M. The loops l_α and l'_α were defined by following a path to the line segment $L_\alpha(\mathbb{R})$, which hopped over any line segment once, and then by following a small loop which encircled $L_\alpha(\mathbb{R})$. Note that $L_\alpha(\mathbb{R})$ intersects any other line segment in at most a double point. Thus, we can homotop the small loop used in the definition of l'_α past any intersection points in $L_\alpha(\mathbb{R})$, into the small loop which was used to define the loop l_α. In this way, we can deform the loop l'_α into a new loop l''_α which is based at p, hops over every line segment on its way to $L_\alpha(\mathbb{R})$, and which follows the same small loop around $L_\alpha(\mathbb{R})$ as the loop l_α. Thus, we are reduced to showing that we are able to homotop the path used to define the loop l''_α into the path which is used to define l_α. This can be done by deforming the path which defines l''_α, over the double points in $\mathscr{A}(\mathbb{R})$, into the path which defines the loop l_α. We are able to do this using Lemma 5.5. □

6. The Generic fiber

In this section we choose a basepoint for the space X_n^2. We then find a natural presentation for the fundamental group of the generic fiber, GF_n^2, of the map $p_n^2 : X_n^2 \to X_{n-1}^2$.

To give us some flexibility later, we begin by finding a contractible subset B of $X_n^2(\mathbb{R})$ which we will use to "fatten the base point". Standard homotopy theory implies that the inclusion $(X_n^2, *) \hookrightarrow (X_n^2, B)$ induces a canonical isomorphism $\pi_1(X_n^2, *) \hookrightarrow \pi_1(X_n^2, B)$ for all $* \in B$. Hence, elements of $\pi_1(X_n^2)$ may be represented by paths whose endpoints lie within B.

Let $x = (x_1, \ldots, x_n)$ be an element of $X_n^2(\mathbb{R})$ which is mapped to the point $(x_1, \ldots, x_{n-1})$ by the map p_n^2. Denote the line in the fiber of p_n^2 over the point x, which passes through x_i and x_j, by L_{ij}. Let $\mathscr{A}_x$ be equal to the union of the lines L_{ij}, $1 \leq i < j \leq n-1$. The fiber of the map p_n^2 over the point $(x_1, \ldots, x_{n-1}) \in X_{n-1}^2$ is then equal to $\mathbb{A}^2 - \mathscr{A}_x$. Choose coordinates for $\mathbb{A}^2$. Define $\psi : \mathbb{C} \to \mathbb{C}^2$, by setting $\psi(t) = (t, t^2)$, i.e. the rational normal curve. Define the set B to be equal to

$$\{(\psi(t_1), \ldots, \psi(t_n)) \mid t_i \in \mathbb{R},\ t_1 \leq \cdots \leq t_n\}.$$

Since the curve $\psi(\mathbb{R})$ is convex, the set B is a subset of $X_n^2(\mathbb{R})$. The set B is clearly contractible.

The fiber p_n^2 over a point in B is not necessarily generic. When choosing a base point in X_n^2 we need to ensure that this is the case and thus we impose two extra conditions on points in B. To do this we first define some new lines in the fiber of p_n^2. Consider the curve $\psi(\mathbb{R})$ as being a subset of in $\mathbb{P}^2(\mathbb{R})$, i.e. the parabola $\psi(\mathbb{R})$ together with an extra point at infinity. Let $L_k^\infty(\mathbb{R})$, $1 \leq k \leq n$, be equal to the line in $\mathbb{R}^2$ which passes through the point x_k and the extra point at infinity determined by $\psi(\mathbb{R})$ (so that the lines $L_k^\infty(\mathbb{R})$, $1 \leq k \leq n$, are parallel to one another). Orient the line

$L_k^\infty(\mathbb{R})$ in the direction pointing to the interior of $\psi(\mathbb{R})$. We now specify the two extra conditions that we impose on points in B;

- Divide the line $L_k^\infty(\mathbb{R})$, $1 \le k \le n-1$, into three segments as follows. Let the first segment be the portion of $L_k^\infty(\mathbb{R})$ contained in the interior of $\psi(\mathbb{R})$. Let the second segment be the portion of the line $L_k^\infty(\mathbb{R})$ between the point $x_k \in L_k^\infty(\mathbb{R})$ and the point $L_{12}(\mathbb{R}) \cap L_k^\infty(\mathbb{R})$. Let the third segment be the remaining portion of $L_k^\infty(\mathbb{R})$. A point x of B satisfies the *lexcigon condition* if it satisfies the following three properties:
    1. For each $k+2 \le j \le n$, the line $L_{ij}(\mathbb{R})$, $1 \le i \le k-1$, intersects the first segment of the line $L_k^\infty(\mathbb{R})$ in a point which lies above the line $L_{i,j-1}(\mathbb{R})$, $1 \le i \le k$.
    2. The lines $L_{ij}(\mathbb{R})$, $1 \le i \le j < k$, intersect the the second segment of the line $L_k^\infty(\mathbb{R})$ in lexcigongraphical order with respect to the orientation of $L_k^\infty(\mathbb{R})$.
    3. The lines $L_{ij}(\mathbb{R})$, $k+1 \le i < j \le n$, intersect the third segment of $L_k^\infty(\mathbb{R})$ in reverse lexcigongraphical order with respect to the orientation of $L_k^\infty(\mathbb{R})$.
- The line $L_k^\infty(\mathbb{R})$, $1 \le k \le n-1$ and the line $L_{kn}(\mathbb{R})$ divide $\mathbb{A}^2(\mathbb{R})$ into four regions. Two of these regions do not contain the curve $\psi(\mathbb{R})$. A point x of B satisfies the *double condition* if the regions not containing the curve $\psi(\mathbb{R})$ do not contain any double points of the arrangement $\mathscr{A}_x$, for all $1 \le k \le n-1$.

Remark 6.1. The lexcigon condition ensures that we do not get parallel lines in the fiber of p_n^2 over x as in Fig. 3. The double condition ensures that we only get double points in $\mathscr{A}_x$ away from the points x_k. Hence, we avoid degenerations in the fiber of p_n^2 like the one illustrated in Fig. 4.

Define the set S_n to be the set of points contained in B which satisfy the lexcigon and triple conditions. Note that $p_n^2(S_n)$ is equal to S_{n-1}.

Lemma 6.2. *The space S_n is homeomorphic to the the space $S_{n-1} \times \mathbb{R}$ for all $n \ge 2$.*

Proof. We proceed by induction on n. When $n=1$ the set S_1 is equal to $\mathbb{R}$. Assume the result up to $n-1$. Let $(x_1,\dots,x_{n-1}) \in S_{n-1}$. Let c be the final point to the right of x_{n-1} on $\psi(\mathbb{R})$, for which $(x_1,\dots,x_{n-1},c)$ fails to satisfy both the lexcigon and double conditions. The set of points, F, to the right of c on $\psi(\mathbb{R})$ is homeomorphic to $\mathbb{R}$. Moreover, the point $(x_1,\dots,x_{n-1},f)$ is in S_n for all $f \in F$. The result follows. □

Corollary 6.3. *The set S_n is contractible for each $n \ge 1$.*

We can now inductively choose a base point for X_n^2. Choose any point in S_1. Let $b=(b_1,\dots,b_n)$ be a fixed point in the set S_n, with b lying over the basepoint previously chosen in S_{n-1}. From now on define b to be the base point of X_n^2, and we consider GF_n^2 as being the (generic) fiber of p_n^2 over this basepoint. For example, see Fig. 9 for a picture of the generic fiber over the point $b \in X_4^2$.

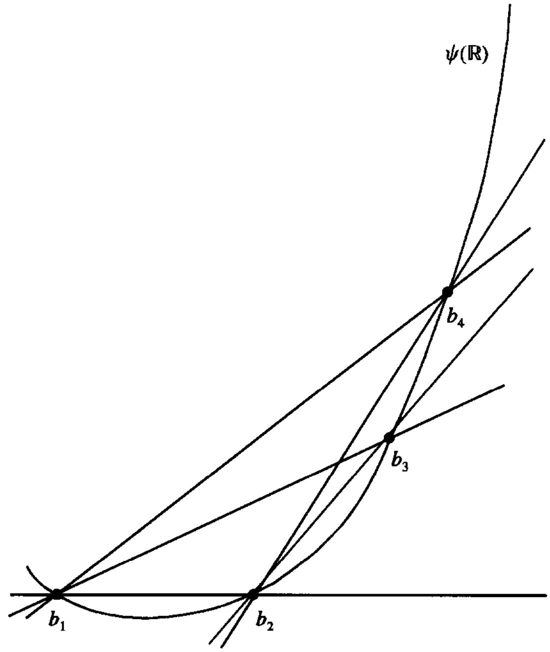


Fig. 9. The generic fiber.

Now, let $p_k^2 : X_n^2 \to X_{n-1}^2$ be the projection which forgets the kth point, for $1 \le k \le n-1$. The set S_n was specifically chosen so that the following result would be true.

Proposition 6.4. *The fiber of the map p_n^2 over any point in S_{n-1} is generic. Moreover, the oriented matroid determined by the real arrangement of lines in the fiber of the map p_k^2, $1 \le k \le n$, over any point in S_{n-1} is isomorphic to that determined by the real arrangement of lines in GF_n^2.*

Let F_k denote the fiber of the map $p_k^2 : X_n^2 \to X_{n-1}^2$ over the a point in S_{n-1}. By [2, Theorem 5.3] we immediately obtain the following result.

Corollary 6.5. *The groups $\pi_1(GF_n^2)$ and $\pi_1(F_k)$ are isomorphic for $1 \le k \le n-1$.*

We shall now find a presentation for the group $\pi_1(GF_n^2, b_n)$, where $GF_n^2 = \mathbb{C}^2 - \mathscr{A}_b$ is the generic fiber over the basepoint $b = (b_1, \ldots, b_{n-1}) \in X_{n-1}$. We work in the neighborhood N_ε defined in Section 5. Note that since N_ε is homotopy equivalent to $\mathbb{C}^2 - \mathscr{A}_b$ the two groups $\pi_1(GF_n^2, b_n)$ and $\pi_1(N_\varepsilon, b_n)$ are isomorphic.

We begin by finding generators. Orient all lines $L_{ij}(\mathbb{R})$, in the direction from b_i to b_j where $i<j$. Let $p_{ij} = L_{ij}(\mathbb{R}) \cap L_n^\infty(\mathbb{R})$. We define loop a_{ijn}, $1 \le i < j \le n-1$, to be the loop in L_n^∞, based at b_n, which goes around p_{ij} (see Fig. 10). As a consequence of Proposition 5.3 we immediately obtain the following result.

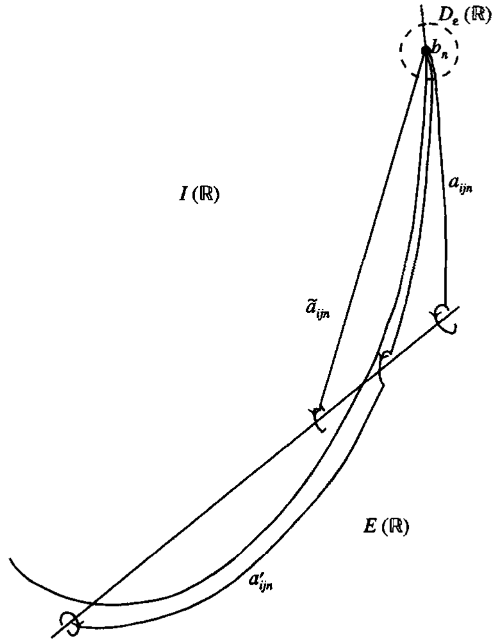


Fig. 10. The loops a_{ijn}, $\tilde{a}_{ijn}$ and a'_{ijn}.

Lemma 6.6. *The set* $\{a_{ijn} \mid 1 \leq i<j \leq n-1\}$, *generates the group* $\pi_1(GF_n^2, b_n)$.

We now wish to find the defining relations amongst the a_{ijn} for the group $\pi_1(GF_n^2, b_n)$. We will do this using Van Kampen's Theorem. We begin by dividing $\mathbb{R}^2 \subset \mathbb{C}^2$ into three open sets. Let $\varepsilon > 0$ be a small real number. Let $T(\mathbb{R})$ be a real tubular neighborhood of the curve $\psi(\mathbb{R})$, with diameter equal to 2ε. Choose ε small enough so that $T(\mathbb{R})$ does not contain any double intersections of $\mathscr{A}_b$. Let $D_\varepsilon(\mathbb{R}) \subset T(\mathbb{R})$ be the disc of radius ε centered at b_n. The curve $\psi(\mathbb{R})$ divides $\mathbb{R}^2$ into two regions. Call the open region containing the tangent lines to $\psi(\mathbb{R})$ the *exterior* region of $\psi(\mathbb{R})$. The complementary open region will be called the *interior* region of $\psi(\mathbb{R})$. Let $E(\mathbb{R})$ denote the union of the exterior region of $\psi(\mathbb{R})$ and the disc $D_\varepsilon(\mathbb{R})$. Let $I(\mathbb{R})$ denote the union of the interior region of $\psi(\mathbb{R})$ and the disc $D_\varepsilon(\mathbb{R})$. Note that $\mathbb{R}^2 = T(\mathbb{R}) \cup E(\mathbb{R}) \cup I(\mathbb{R})$. Let E, I and T denote the complexification of each of these sets, respectively, intersected with N_ε.

Our aim is to find a presentation for $\pi_1(GF_n^2, b_n)$ by applying Van Kampen's Theorem to the sets $I - \mathscr{A}_b$, $E - \mathscr{A}_b$, and $T - \mathscr{A}_b$. To do this, we begin by finding a presentation of $\pi_1(I - \mathscr{A}_b, b_n)$. We first need to define a new loop $\tilde{a}_{ijn}$, $1 \leq i<j \leq n-1$, in GF_n^2. Define the loop $\tilde{a}_{ijn}$ as follows. Pick a point $p_{ij} \in I(\mathbb{R})$ on the line $L_{ij}(\mathbb{R})$ which lies within the disc of radius ε about the point b_j. Let the line $\tilde{L}(\mathbb{R})$ denote the real line joining b_n and p_{ij}. We define the loop $\tilde{a}_{ijn}$ to be the loop in $\tilde{L}$, based at b_n, which goes around p_{ij} (see Fig. 10).

Lemma 6.7. *The group* $\pi_1(I - \mathscr{A}_b, b_n)$ *admits a presentation with generators*

$$\tilde{a}_{ijn}, \quad 1 \le i < j \le n-1,$$

and defining relations

$$[\tilde{a}_{ijn}, \tilde{a}_{rsn}] = 1, \quad 1 \le i < r < j < s \le n-1. \tag{6}$$

Proof. Note that the set $I(\mathbb{R})$ is convex. Hence, the lines $L_{ij}(\mathbb{R})$ intersect $I(\mathbb{R})$ in only one segment. To complete the proof apply Lemma 5.6 to the set I. □

We now find a presentation for the group $\pi_1(E - \mathscr{A}_b, b_n)$. We begin by defining a new loop a'_{ijn}, $1 \le i < j \le n-1$, contained in GF_n^2. Let $\psi'(\mathbb{R})$ be a curve joining b_n and b_{i-1} which is obtained by deforming the curve $\psi(\mathbb{R})$ as follows. Fix the points b_n and b_{i-1} and push the portion of the curve $\psi(\mathbb{R})$ lying between these two points away from the curve $\psi(\mathbb{R})$ into the region $E(\mathbb{R})$. Let p_{rs} be equal to $L_{rs}(\mathbb{R}) \cap \psi'(\mathbb{R})$, $1 \le r < s \le n-1$. We now define the loop a'_{ijn}. Start at the point b_n. On reaching a point p_{rs} hop over the line $L_{rs}(\mathbb{R})$. Run all the way down $\psi'(\mathbb{R})$, hopping over each point p_{rs}, until reaching the line $L_{ij}(\mathbb{R})$. Then choose a line $L(\mathbb{R})$ passing through p_{ij} which is tranverse to $L_{ij}(\mathbb{R})$. Let l be a small loop in L, which is oriented in the positive direction with respect to the orientation of L, and which goes around L_{ij}. Run around the loop l in the positive direction. Finally, return to b_n along the same path taken on the outward journey.

Lemma 6.8. *The group* $\pi_1(E - \mathscr{A}_b, b_n)$ *admits a presentation with generators*

$$a_{ijn} \text{ and } a'_{ijn} \quad 1 \le i < j \le n-1$$

and defining relations

$$[a_{ijn}, a_{rsn}] = 1, \quad 1 \le r < i < j < s \le n-1, \tag{7}$$

$$[a_{ijn}, a'_{rsn}] = 1, \quad 1 \le i < j < r < s \le n-1. \tag{8}$$

Proof. Note that the set $E(\mathbb{R})$ is not convex, and that, in fact, the intersection of each line $L_{ij}(\mathbb{R})$ with $E(\mathbb{R})$ consists of precisely two line segments. Now, apply Lemma 5.6, to the set E, noting that for each line $L_{ij}(\mathbb{R})$ we have to add the two generators a_{ijn} and a'_{ijn} to the presentation of $\pi_1(E - \mathscr{A}_b, b_n))$, corresponding to the two line segments. □

Now, we find a presentation for the group $\pi_1(T - \mathscr{A}_b, b_n)$. Let $[g_1, \ldots, g_n]$ denote the set of relations

$$g_1 \ldots g_n = g_2 \ldots g_n g_1 = \cdots = g_n g_1 \ldots = g_{n-1}.$$

Lemma 6.9. *The group* $\pi_1(T - \mathscr{A}_b, b_n)$ *admits a presentation with generators*

$$a_{ijk}, \quad \tilde{a}_{ijn}, \quad 1 \le i < j \le n-1,$$

and defining relations

$$[a_{1jn},\dots,a_{j-1,jn},\tilde{a}_{j,j+1,n},\dots,\tilde{a}_{j,n-1,n}]=1, \quad 1\le j\le n-1. \tag{9}$$

Proof. The generators a_{ijn} and $\tilde{a}_{ijn}$ can be homotoped into the set $T-\mathscr{A}_b$. To see this deform the paths that were used to define the loops a_{ijn} and $\tilde{a}_{ijn}$ into the set $T-\mathscr{A}_b$. This can be done by deforming these paths over the double points in $\mathscr{A}_b(\mathbb{R})$, using Lemma 5.5. Then deform the small loops in the definition of the a_{ijn} and a'_{ijn}, past the intersection points in lines $L_{ij}(\mathbb{R})$, until they lie in the set $T-\mathscr{A}_b$.

By making ε small enough we see that

$$\pi_1(T-\mathscr{A}_b,b_n)\cong \coprod_j \pi_1\left(B_\varepsilon(b_j)-\bigcup_i L_{ij}\right),$$

where $B_\varepsilon(b_i)$ is a complex ball or radius ε with center b_j. We now refer to Randell's paper [16]. By considering the ends of the loops a_{ijk} and $\tilde{a}_{ijk}$ locally in each ball $B_\varepsilon(b_j)$ we see that the relations (9) arise from a calculation made within the complement of the Hopf link, $\delta(B_\varepsilon(b_j))\cap\bigcup L_{ij}$. □

We now want to write the loops $\tilde{a}_{ijn}$ and a'_{ijn} in terms of the generators a_{ijk}. We refer to Randell's paper [16]. Using the result contained in this paper concerning the Hopf link of a point, we have the formulas

$$\tilde{a}_{ijn}=A^{-1}a_{ijn}A, \tag{10}$$

where,

$$A=a_{i-1,jn}\dots a_{1jn} \tag{11}$$

and

$$a'_{ijn}=B^{-1}\,\tilde{a}_{ijn}B, \tag{12}$$

where

$$B=\tilde{a}_{i,j-1,n}\dots\tilde{a}_{i,i+1,n}a_{i,i-1,n}\dots a_{1,i,n}. \tag{13}$$

We are now able to state the main theorem of this section.

Theorem 6.10. *The group $\pi_1(GF_n^2,b_n)$ admits a presentation with generators*

$$a_{ijn}, \quad 1\le i<j\le n-1$$

and defining relations (6)–(9).

Proof. As we have seen, $N_\varepsilon\simeq\mathbb{C}^2-\mathscr{A}_b$ can be divided into three sets, $E-\mathscr{A}_b, I-\mathscr{A}_b$ and $T-\mathscr{A}_b$. We know a presentation for the fundamental group of each of these sets. Hence, we need only understand how they fit together. The fundamental group of the intersection $E \cap T$ is a free group with generators a_{ijn} and a'_{ijn}. The fundamental group of the intersection $I \cap T$ is a free group with generators $\tilde{a}_{ijn}$. Also, $E \cap I = D_\varepsilon$ which is contractible. Now apply Van Kampen's Theorem. □

In Section 9 we will need to know a presentation for the group $\pi_1(\mathbb{P}^2 - \mathscr{A}_b, b_n)$, which arises as the fundamental group of the generic fiber of the projection $q_n^2 : Y_n^2 \to Y_{n-1}^2$.

Lemma 6.11. *The group $\pi_1(\mathbb{P}^2 - \mathscr{A}_b, b_n)$ admits a presentation with the same generators and relations as $\pi(GF_n^2, b_n)$, and with the additional relation*

$$a_{12n}\, a_{13n} \ldots a_{1,n-1,n} a_{23n} \ldots a_{2,n-1,n} a_{34n} \ldots a_{n-2,n-1,n} = 1. \tag{14}$$

Proof. Use Van Kampen's theorem to "glue" the line at infinity into GF_n^2. Note that this line is homeomorphic to a punctured copy of $\mathbb{P}^1$: hence the relation (14). □

7. Main theorems

We begin this section by finding generators for the group P_n^2. Let F_k be equal to the fiber[3] over the base point $b \in S_{n-1}$, of the projection $X_n^2 \to X_{n-1}^2$, obtained by forgetting the kth point, for $1 \le k \le n$. Let $L(\mathbb{R})$ be the tangent line to $\psi(\mathbb{R})$ passing through the point b_k. Define loop a_{ijk}, $1 \le i < j < k \le n$, to be the loop in L, based at b_k, which goes around $L(\mathbb{R}) \cap L_{ij}(\mathbb{R})$ (see Fig. 11). Define loops a'_{ijk} and $\tilde{a}_{ijk}$ in P_n^2 using formulas (10)–(13) with $n = k$.

Lemma 7.1. *The group P_n^2, $n \ge 3$, is generated by the set*

$$\{a_{ijk} \mid 1 \le i < j < k \le n\}.$$

Proof. We proceed by induction. The group P_3^2 isomorphic to $\mathbb{Z}$ by Proposition 3.4, thus when $n = 3$ the result is clear. Assume the result up to $n - 1$. Now use sequence (5). The generators a_{ijk}, where $1 \le i < j < k \le n-1$, generate P_{n-1}^2 by induction, and clearly lift from P_{n-1}^2 to P_n^2. By adding the the generators a_{ijn} we obtain the result. □

We now want to find relations amongst the a_{ijk} in order to find a presentation for the group P_n^2. We denote the lexcigongraphical ordering on the set of two element subsets of $\{1, \ldots, n\}$ by $\prec$.

Theorem 7.2. *The following relations hold in P_n^2 for $3 \le k \le n$:*

$$[a_{ijk}, a_{rsk}] = 1, \quad 1 \le r < i < j < s \le k, \tag{15}$$

$$[a_{ijk}, a'_{rsk}] = 1, \quad 1 \le i < j < r < s \le k, \tag{16}$$

[3] Note that when k is equal to n then F_k is equal to GF_n^2.

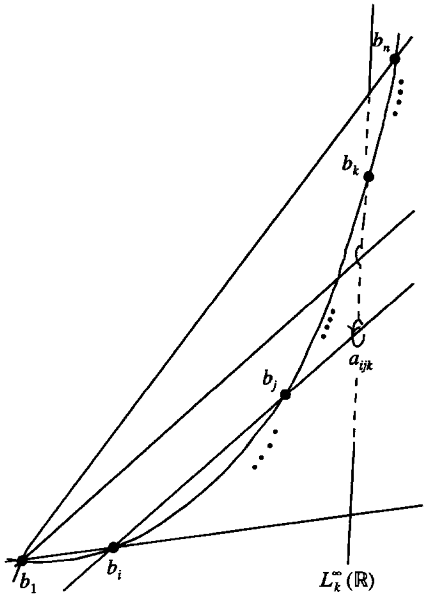


Fig. 11. The generator a_{ijk} of P_n^2.

$$[\tilde{a}_{ijk}, \tilde{a}_{rsk}] = 1, \quad 1 \le i < r < j < s \le k, \tag{17}$$

$$[a_{1jk}, \ldots, a_{j-1,jk}, \tilde{a}_{j,j+1,k}, \ldots, \tilde{a}_{j,k-1,k}, a_{jk,k+1}, \ldots, a_{jk,n-1}] = 1, \quad 1 \le j \le k-1, \tag{18}$$

$$a_{ijk}^{-1} a_{rst} a_{ijk} \tag{19}$$

$$\begin{cases} a_{rst} & rs \prec ij \text{ or } jk \prec st, \\ a_{ijt}\, a_{ikt}\, a_{rst}\, a_{ikt}^{-1}\, a_{ijt}^{-1} & jk = rs, \\ a_{ijt}\, a_{ikt}\, a_{jkt}\, a_{ikt}^{-1}\, a_{ijt}^{-1}\, a_{jkt}^{-1}\, a_{rst}\, a_{jkt}\, a_{ijt}\, a_{ikt}\, a_{jkt}^{-1}\, a_{ikt}^{-1}\, a_{ijt}^{-1} & ik \prec rs \prec jk, \\ a_{ijt}\, a_{ikt}\, a_{jkt}\, a_{rst}\, a_{jkt}^{-1}\, a_{ikt}^{-1}\, a_{ijt}^{-1} & ik = rs \text{ or } ij = rs, \\ a_{ijt}\, a_{ikt}\, a_{jkt}\, a_{ijt}^{-1}\, a_{jkt}^{-1}\, a_{ikt}^{-1}\, a_{rst}\, a_{ikt}\, a_{jkt}\, a_{ijt}\, a_{jkt}^{-1}\, a_{ikt}^{-1}\, a_{ijt}^{-1} & ij \prec rs \prec ik, \text{ where } k < t \le n. \end{cases}$$

Proof. This proof proceeds inductively using sequence (5). When $n = 3$ the relations degenerate, which is correct since P_3^2 is isomorphic to $\mathbb{Z}$.

At the nth stage we split the proof up into three main parts and deal with each separately in the following sections:

1. When $k = n$ relations (15)–(18) are simply those coming from the fiber group. These were established in Section 6.

2. When $t = n$ relations (19) come from conjugation of generators in the fiber group by the generators of P_n^2. These will be established in Section 10.

3. Relations not dealt with in (1) or (2) come from lifting relations from P_{n-1}^2. These will be established in Section 12. □

If sequence (5) were short exact, then Theorem 7.2 would give us precisely the relations required for a presentation of P_n^2. This motivates the following definition.

Definition 7.3. Let PL_n be the group whose presentation has generators a_{ijk}, $1 \le i < j < k \le n$, and relations (15)–(19) of Theorem 7.2.

Let $\varphi_n : PL_n \to P_n^2$ be the tautological homomorphism which takes generators to generators. Then, φ_n is clearly well defined and onto.

We now want to see how close the group PL_n is to being isomorphic to P_n^2. We will do this by studying the relationship between the integral homology of each of these groups. Specifically, we will show that φ_n induces an isomorphism on the first two integral homology groups. First, we will need some preliminary results concerning the homology of the space X_n^2.

Lemma 7.4. *The restriction map*

$$H^k(X_n^2, \mathbb{Z}) \to H^k(GF_n^2, \mathbb{Z})$$

is surjective.

Proof. The group $H^1(X_n^2, \mathbb{Z})$ contains the classes of the forms $(1/2\pi i)(\mathrm{d}\Delta_{ijk}/\Delta_{ijk})$, where Δ_{ijk} are the minors defined in the introduction. The forms $(1/2\pi i)(\mathrm{d}\Delta_{ijn}/\Delta_{ijn})$ restrict to the forms $(1/2\pi i)(\mathrm{d}L_{ijn}/L_{ijn})$ on GF_n^2 whose k-fold wedge products generate $H^k(GF_n^2, \mathbb{Z})$ [3]. $\square$

Corollary 7.5. *The group $H_k(GF_n^2, \mathbb{Z})$ injects into $H_k(X_n^2, \mathbb{Z})$ for all $k \ge 0$.*

Proposition 7.6. *The Leray spectral sequence in the homology for the map $p_n^2 : X_n^2 \to X_{n-1}^2$ has the following properties (here all homology groups have $\mathbb{Z}$ coefficients, unless otherwise stated):*

1. *for $p+q \le 2$ the $E^2_{p,q}$ terms are isomorphic to*

$H_2(GF_n^2)$	$*$	$*$
$H_1(GF_n^2)$	$H_1(GF_n^2) \otimes H_1(X_{n-1}^2)$	$*$
$\mathbb{Z}$	$H_1(X_{n-1}^2)$	$H_2(X_{n-1}^2)$

;

2. *let $p+q \le 2$. Then the d_2 differentials whose images lie in $E^2_{p,q}$ all vanish, and thus $E^2_{p,q} = E^\infty_{p,q}$;*
3. *the terms $E^2_{p,q}$ are torsion free for $p+q \le 2$.*

Proof. We begin by proving statement (1). We will work with the cohomology spectral sequence first, and justify the statement in homology later. Let π denote the projection map $p_n^2 : X_n^2 \to X_{n-1}^2$. Let $\mathscr{U}$ be an open cover of X_{n-1}^2. Let $\mathscr{F}_q$ be the sheaf defined by $\mathscr{F}_q(U) = H^q(\pi^{-1}(U), \mathbb{Z})$, where $U \in \mathscr{U}$. Then the E_2 term of the Leray spectral sequence in cohomology is given by

$$E_2^{p,q} = H^p(X_n^2, \mathscr{F}_q)$$

and converges to $H^{p+q}(X_n^2, \mathbb{Z})$. If the map π were a fibration then $\mathscr{F}_q$ would be locally constant for each q and we would obtain the result, since the cohomology groups of the fiber of π are torsion free abelian groups of finite rank [3]. However, this is not the case. To obtain the result we show that $\mathscr{F}_q$ is locally constant for $q = 1, 2$.

Since the projection p_n^2 is a fibration over the complement of the discriminant locus with fiber a hyperplane complement, it follows that away from the discriminant locus, each of the sheaves $\mathscr{F}_q$ is a local system of torsion free abelian groups. Moreover, since the hyperplanes are labelled, the local system of H^1's is trivial over the complement of the discriminant locus. Since the cohomology ring of a hyperplane complement is generated by H^1, it follows that each $\mathscr{F}_q$ is a trivial local system of torsion free abelian groups over the complement of the discriminant locus. Hence, since the fundamental group of X_{n-1}^2 less the discriminant locus surjects onto the fundamental group of X_{n-1}^2, it follows that $\mathscr{F}_1$ and $\mathscr{F}_2$ are trivial as local systems, once we prove that they are local systems.

Let x be any point in X_{n-1}^2. We will show that we can choose an open ball $B_x \subset X_{n-1}^2$, containing x, so that the group $H^q(\pi^{-1}(B_x), \mathbb{Z})$ is isomorphic to $H^q(GF_n^2, \mathbb{Z})$ for $q = 1, 2$.

Denote the complex dimension of the space X_{n-1}^2 by m. The fiber over the point x in X_{n-1}^2 is equal to $\mathbb{C}^2$ less a union of lines L_{ij}, $1 \le i < j \le n-1$. Let $D_{ijn} \subset (\mathbb{C}^2)^n$ denote the divisor defined by the minor Δ_{ijn} defined in Section 1. Then we choose an open ball B_x containing x so that $D_{ijn} \cap \pi^{-1}(B_x)$ is homeomorphic to $L_{ij} \times B_x$. Denote the set $L_{ij} \times B_x$ by F_{ij}. Then $\pi^{-1}(B_x)$ is homeomorphic to a complex ball M of dimension $m+2$ minus the union of the F_{ij}. Since the intersection lattices of $M - \bigcup F_{ij}$ and GF_n^2 agree in complex codimensions 1 and 2, the groups $H^q(\pi^{-1}(B_x), \mathbb{Z})$ and $H^q(GF_n^2, \mathbb{Z})$ are isomorphic for $q = 1, 2$ (for a detailed proof of this fact see Example 1, and Corollaries 3 and 4 of [10]).

We prove (2) and (3) together, using induction. To begin the induction, note that $H_0(X_3^2) = \mathbb{Z}$, $H_1(X_3^2) = \mathbb{Z}$, and $H_2(X_3^2) = 0$. Since GF_n^2 is a complement of lines in $\mathbb{C}^2$, we know that $H_k(GF_n^2)$ (and $H^k(GF_n^2)$) is torsion free for all $k \ge 0$ [3]. We work first with the Leray spectral sequence in cohomology with rational coefficients. Assume inductively that $H^k(X_{n-1}^2)$ is torsion free for $k = 0, 1$ and 2. From Lemma 7.4 we deduce that all of the differentials $d_2 : E_2^{0,q} \to E_2^{2,q-1}$ vanish. By multiplicativity of the spectral sequence in cohomology, this implies that the differential $d_2 : E_2^{1,1} \to E_2^{3,0}$ also vanishes.

Now, dualize to consider the homology spectral sequence. Note that all of the required differentials vanish when the terms of the spectral sequence have $\mathbb{Q}$ coefficients. However, since the fiber homology is torsion free all of the differentials whose image lie in the fiber homology groups vanish when the terms have $\mathbb{Z}$ coefficients. The only other differential which could be non-zero is $d^2 : E_{3,0}^2 \to E_{1,1}^2$. However, this has to vanish since $H_1(GF_n^2) \otimes H_1(X_{n-1}^2)$ is torsion free by induction. $\square$

Corollary 7.7. *The first and second integral homology groups of* X_n^2 *are torsion free.*

Proof. Since the Leray spectral sequence in homology degenerates at $E^2_{p,q}$, when $p+q \leq 2$, the graded quotients of the corresponding filtration of $H_1(X_n^2,\mathbb{Z})$ and $H_2(X_n^2,\mathbb{Z})$ are torsion free, from which the result follows. □

Remark 7.8. The E_∞ term of the Leray spectral sequence in cohomology induces a decreasing filtration $L^\bullet$ on $H^\bullet(X_n^2,\mathbb{Z})$ which we shall call the Leray filtration.

We will also need a technical corollary.

Corollary 7.9. *The cup product* $\Lambda^2 H^1(X_n^2,\mathbb{Z}) \to H^2(X_n^2,\mathbb{Z})$ *is surjective.*

Proof. As a graded ring, $E_\infty^{p,q}$ is isomorphic to $Gr_L^p H^{p+q}(X_n^2)$, where Gr_L^p denotes the pth graded quotient of the filtration $L^\bullet$. From the proof of Proposition 7.6 we deduce that $\Lambda^2 Gr_L^\bullet H^1(X_n^2,\mathbb{Z})$ surjects onto $Gr_L^\bullet H^2(X_n^2,\mathbb{Z})$, which implies the result by induction on n. □

Another immediate consequence of Proposition 7.6 is the following result.

Corollary 7.10. *The sequence*

$$0 \to H_1(GF_n^2,\mathbb{Z}) \to H_1(X_n^2,\mathbb{Z}) \to H_1(X_{n-1}^2,\mathbb{Z}) \to 0$$

is short exact.

Proof. This sequence arises from the filtration of $H_1(X_n^2,\mathbb{Z})$ given by the E^∞ term of the Leray spectral sequence, plus the fact that $H_1(GF_n^2,\mathbb{Z})$ injects into $H_1(X_n^2,\mathbb{Z})$ by Corollary 7.5. □

This corollary immediately gives us the following lemma.

Lemma 7.11. *The group* $H_1(X_n^2,\mathbb{Z})$ *is free abelian group of rank* $\binom{n}{3}$.

Proof. Proceed by induction. To begin the induction note that the group $H_1(X_3^2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, since $\pi_1(X_3^2)$ equal to $\mathbb{Z}$ (Lemma 3.3).

Since GF_n^2 is the complement in $\mathbb{A}^2$ of $\binom{n-1}{2}$ lines, the group $H_1(GF_n^2,\mathbb{Z})$ is free abelian of rank $\binom{n-1}{2}$. By induction the group $H_1(X_{n-1}^2,\mathbb{Z})$ is free abelian of rank $\binom{n-1}{3}$. The result follows from Corollary 7.10 and the fact that

$$\binom{n-1}{3} + \binom{n-1}{2} = \binom{n}{3}. \quad \square$$

Since we do not know if X_n^2 is an Eilenberg–MacLane space, it is necessary to prove that the first and second integral homology groups of X_n^2 are equal to those of

P_n^2. To do this we will use some homotopy theory. Denote the Eilenberg–MacLane space $K(G,1)$ associated to G by BG.

Lemma 7.12. *If $(X,*)$ is a connected, pointed topological space with the homotopy type of a CW complex and fundamental group G, then the following statements hold:*

1. *there is a natural map $(X,*)\to(BG,*)$ which is unique up to homotopy;*
2. *the homotopy fiber U of the map in 1. is weakly homotopy equivalent to the universal cover of X.*

Proof. This is a standard result in homotopy theory. For the proof of (1) see [23, Corollary 2.4, p. 218]. The proof of (2) follows from the definition and universal mapping properties of the universal cover. □

Proposition 7.13. *When $k=1,2$ the natural map*

$$H_k(X_n^2,\mathbb{Z})\to H_k(P_n^2,\mathbb{Z})$$

is an isomorphism.

Proof. In this proof all homology groups have integer coefficients, and we denote $\pi_1(X_n^2)$ by G. Using Hurewicz's Theorem, we see that the result is true for $k=1$.

By Lemma 7.9, $H_2(X_n^2)$ is generated by cup products. Thus, since

$$\begin{array}{ccc} \Lambda^2 H^1(G) & = & \Lambda^2 H^1(X_n^2) \\ \downarrow & & \downarrow \\ H^2(G) & \xrightarrow{i^*} & H^2(X_n^2) \end{array}$$

commutes, it follows that $H^2(G)\to H^2(X_n^2)$ is surjective.

By Lemma 7.12 we know that there is a natural map

$$(X_n^2,*)\to(BG,*)$$

whose homotopy fiber, U_n^2, is weakly homotopy equivalent the universal cover of X_n^2. Since U_n^2 is simply connected, the E_2 term Leray–Serre spectral sequence in cohomology of this fibration is

$H_2(U_n^2)$	$H_2(G,H_1(U_n^2))$	$H_2(G,H_2(U_n^2))$
0	0	0
$\mathbb{Z}$	$H_1(G)$	$H_2(G)$

Thus, it follows that $H^2(G)\to H^2(X_n^2)$ is injective. Hence, $H^2(G)$ is isomorphic to $H^2(X_n^2)$, and therefore, by the universal coefficient theorem, $H_2(X_n^2,\mathbb{Q})$ is isomorphic to $H_2(G,\mathbb{Q})$. Since $H_2(X_n^2)$ is torsion free by Corollary 7.7, this implies that $H_2(X_n^2)\to H_2(G)$ is injective.

Finally, using the Leray spectral sequence in homology of the map $(X_n^2,*)\to(BG,*)$, we also see that $H_2(X_n^2)\to H_2(G)$ is surjective (as $d_2:E^2_{2,0}\to E^2_{0,1}$ vanishes, since the first row is zero), which completes the proof. □

We can now prove the main theorems.

Theorem 7.14. *The homomorphism φ_n induces an isomorphism between the first integral homology groups of PL_n and P_n^2.*

Proof. First, consider the group $H_1(PL_n,\mathbb{Z})$. Since all of the relations in PL_n are commutators, $H_1(PL_n,\mathbb{Z})$ is a free group of rank $\binom{n}{3}$ generated by the homology classes of the elements a_{ijk}, $i\le i<j<k\le n$, which we also denote by a_{ijk}. The map $PL_n\to PL_{n-1}$ induces a map from $H_1(PL_n,\mathbb{Z})\to H_1(PL_{n-1},\mathbb{Z})$. By considering the abelianisation of the relevant groups we obtain the short exact sequence of torsion-free abelian groups

$$0\to K\to H_1(PL_n,\mathbb{Z})\to H_1(PL_{n-1},\mathbb{Z})\to 0,$$

where K is defined to be the kernel. The group K is generated by the homology classes of the elements a_{ijn}, $1\le i<j\le n-1$.

The map φ_n induces the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & H_1(PL_n,\mathbb{Z}) & \longrightarrow & H_1(PL_{n-1},\mathbb{Z}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \varphi_{n*}} & & \downarrow{\scriptstyle \varphi_{n-1*}} & & \\
0 & \longrightarrow & H_1(GF_n^2,\mathbb{Z}) & \longrightarrow & H_1(P_n^2,\mathbb{Z}) & \longrightarrow & H_1(P_{n-1}^2,\mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

The map i is clearly well defined and the bottom row of the diagram is exact by Lemma 7.10. The map i is surjective since $H_1(GF_n^2,\mathbb{Z})$ is freely generated by the homology classes of the loops defined in the presentation of $\pi_1(GF_n^2,b)$, which we denote by γ_{ijn}, $1\le i<j\le n-1$. By definition, the map φ_* maps the class a_{ijn} to the class γ_{ijn} for all $1\le i<j\le n-1$. Since the groups K and $H_1(GF_n^2,\mathbb{Z})$ are both torsion-free abelian groups the map i is an isomorphism.

We now proceed by induction. Note that $H_1(PL_3,\mathbb{Z})$ and $H_1(P_3^2,\mathbb{Z})$ are both isomorphic to $\mathbb{Z}$. Assume by induction that the map φ_{n-1_*} is an isomorphism. The map i is an isomorphism and so we complete the proof by applying the Five Lemma to the above commutative diagram. □

To prove that $H_2(P_n^2,\mathbb{Z})$ is isomorphic to $H_2(PL_n,\mathbb{Z})$ we will use some rational homotopy theory. Let $D:H_2(X,\mathbb{Q})\to\Lambda^2H_1(X,\mathbb{Q})$ be the map induced by the diagonal inclusion $X \hookrightarrow X \times X$. Given a group G, let $\Gamma^n G$ denote the nth term in the lower central series of G.

Lemma 7.15. *If G is any group then the commutator map*

$$\Lambda^2 H_1(G,\mathbb{Z}) \xrightarrow{[\,,\,]} \Gamma^2 G/\Gamma^3 G,$$

defined by

$$x \wedge y \mapsto \overline{xyx^{-1}y^{-1}}$$

is a surjection.

Proof. The group $\Gamma^2 G/\Gamma^3 G$ is abelian, and admits a surjection from the group $H_1(G,\mathbb{Z}) \times H_1(G,\mathbb{Z})$. The proof now follows by inspection. □

The following result is originally due to Sullivan [20], and can be proved using results in either [4, Section 2.1] or [21, Section 8]. The proof is omitted as it is technical (although relatively straightforward) and would be too large a diversion for this paper.

Lemma 7.16. *If X is a topological space and the dimension of $H_1(X,\mathbb{Q})$ is finite for $k=1,2$, then the sequence*

$$H_2(X,\mathbb{Q}) \xrightarrow{D} \Lambda^2 H_1(X,\mathbb{Q}) \xrightarrow{[\,,\,]} [\Gamma^2\pi_1(X)/\Gamma^3\pi_1(X)] \otimes \mathbb{Q} \to 0$$

is exact and natural in X.

Corollary 7.17. *The sequence*

$$0 \to H_2(X_n^2,\mathbb{Q}) \to \Lambda^2 H_1(X_n^2,\mathbb{Q}) \to [\Gamma^2\pi_1(X_n^2)/\Gamma^3\pi_1(X_n^2)] \otimes \mathbb{Q} \to 0$$

is short exact.

Proof. Note that the map $H_2(X,\mathbb{Q}) \to \Lambda^2 H_1(X,\mathbb{Q})$ in Lemma 7.16 is injective if and only if $\Lambda^2 H^1(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$ is surjective. Now apply Lemma 7.9. □

We need to compute the map $D: H_2(PL_n,\mathbb{Q}) \to \Lambda^2 H_1(PL_n,\mathbb{Q})$.

Lemma 7.18. *If F is a free group then*

$$\Lambda^2 H_1(F,\mathbb{Z}) \xrightarrow{[\,,\,]} \Gamma^2 F/\Gamma^3 F$$

is an isomorphism of $\mathbb{Z}$-modules.

Proof. The associated graded group of the descending central series for the free group on a finite set X is isomorphic to the free Lie algebra on X [19, Theorem 6.1, p. 24]. The proof follows easily from this fact. □

Let G be the group

$$G = \langle x_1, \ldots, x_n \mid r \rangle,$$

where $r \in \Gamma^2\langle x_1, \ldots, x_n \rangle$. The relation r gives an element of $H_2(G, \mathbb{Z})$. We will call $D(r)$ the linearization of r. As a consequence of Lemmas 7.18 and 7.16 we have the following result.

Lemma 7.19. *If r is as above then the following statements hold:*

1. *The group $H_2(G, \mathbb{Z})$ is isomorphic to the free abelian group generated by r.*
2. *If $r \equiv \prod [x_i, x_j]^{m_{ij}} \bmod \Gamma^3 G$, then the map D is given by the formula*

$$D(r) = \sum m_{ij}(x_i \wedge x_j).$$

Theorem 7.20. *The homomorphism φ_n induces an isomorphism between the second integral homology groups of PL_n and P_n^2.*

Proof. Let N be the number of relations in PL_n. The rank of $H_2(PL_n, \mathbb{Z})$ is less than or equal to N. When equality occurs $H_2(PL_n, \mathbb{Z})$ is torsion free. So if we can show that the rank of $H_2(PL_n, \mathbb{Z})$ is equal to the dimension of $H_2(PL_n, \mathbb{Q})$, then $H_2(PL_n, \mathbb{Z})$ is torsion free. Thus, we first use rational coefficients.

We begin by defining filtrations on $H_1(PL_n)$, $\Lambda^2 H_1(PL_n)$ and $H_2(PL_n)$. Let G_0 be equal to the set of a_{ijn} where $1 \le i < j \le n-1$, and G_1 be equal to $H_1(PL_n)$. Then the subspaces $\mathrm{span}\{G_i\}$ filter $H_1(PL_n)$. This filtration induces the following filtration on $\Lambda^2 H_1$. Let $F_0 = \mathrm{span}\{a_{ijn} \wedge a_{rsn}\}$. Let $F_1 = \mathrm{span}\{a_{ijk} \wedge a_{rsn}\} \cup F_0$ where $k \neq n$. Finally, let $F_2 = \Lambda^2 H_1(PL_n)$.

To define a filtration on $H_2(PL_n)$ we begin by filtering the relations of PL_n. Let R_0 be the set of relations (15)–(18) with $k = n$. These relations arise from the fiber group. Let R_1 be the set of relations (19) with $t = n$ together with the set relations R_0. The set of relations R_1–R_0 come from conjugating generators in the fiber group by generators in P_{n-1}^2. Finally, let R_2 be the set of all relations for PL_n. We now define a filtration $\tilde{L}$ on $H_2(PL_n)$. Let $\tilde{L}_i$ be the span of the elements of $H_2(PL_n)$ coming from the elements of R_i.

We now proceed with the proof using induction. First, note that both $H_2(PL_3, \mathbb{Z})$ and $H_2(P_n^2, \mathbb{Z})$ are trivial. Assume the result up to $n-1$. As a consequence of Lemma 7.16 we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & H_2(PL_n) & \longrightarrow & \Lambda^2 H_1(PL_n) & \longrightarrow & [\Gamma^2 PL_n / \Gamma^3 PL_n] \otimes \mathbb{Q} & \longrightarrow & 0 \\
 & & \downarrow \varphi_{n*} & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & H_2(X_n^2) & \longrightarrow & \Lambda^2 H_1(X_n^2) & \longrightarrow & [\Gamma^2 P_n^2 / \Gamma^3 P_n^2] \otimes \mathbb{Q} & \longrightarrow & 0.
\end{array}$$

First, we show that the map $D : H_2(PL_n) \to \Lambda^2 H_1(PL_n)$ is injective. The map D is filtration preserving. Thus, the map $\gamma_i : \tilde{L}_i/\tilde{L}_{i-1} \to F_i/F_{i-1}$ induced by D is well defined. To show that D is injective it is sufficient to show that γ_i is injective for $0 \le i \le 2$. We begin with γ_0. The image under D of the fiber relations are

$$a_{ijn} \wedge a_{rsn} \quad \text{if } ij \ne rs, \tag{20}$$

$$a_{ijn} \wedge (a_{1in} + a_{2in} + \cdots + a_{ijn} + \cdots + a_{i,n-1,n}) \quad \text{for } 1 \le i \le n-1. \tag{21}$$

All of these are linearly independent in F_0. Hence, $\tilde{L}_0$ injects into F_0.

Now, consider the map γ_1. The linearized conjugation relations are

$$a_{ijk} \wedge a_{rsn} \quad \text{if } rs \ne ij,\ ik \text{ or } jk, \tag{22}$$

$$a_{rsn} \wedge (a_{ijk} + a_{ijn} + a_{ikn} + a_{ikn}) \quad \text{if } rs = ij,\ ik \text{ or } jk. \tag{23}$$

We now quotient out by F_0. Relations (22) and (23) modulo F_0 become

$$a_{rsn} \wedge a_{ijk}, \quad \text{where } k \ne n. \tag{24}$$

These are linearly independent. Hence γ_1 is injective. The map γ_2 is injective by induction. Hence the map D is injective.

We now show that the map φ_n induces an isomorphism between $H_2(PL_n)$ and $H_2(X_n^2)$. Let L be the filtration on $H_\bullet(X_n^2)$ induced by the Leray sequence. Since the map D is injective for both $H_2(PL_n)$ and $H_2(P_n^2)$ we will not differentiate between H_2 and its image in $\Lambda^2 H_1$, for example, $D(L_0) = L_0$. We show that the map $\beta_i : \tilde{L}_i/\tilde{L}_{i-1} \to L_i/L_{i-1}$ induced by φ_n is an isomorphism for $0 \le i \le 2$.

Begin with the map β_0. Note that $L_0 = H_2(GF_n^2)$. Thus we have to analyze the group $H_2(GF_n^2)$. Recall that $GF_n^2 = \mathbb{C}^2 - \mathscr{A}_b$. Let p be an intersection point in $\mathscr{A}_b$ and $\mathscr{A}_p$ be the set of lines in $\mathscr{A}$ passing through the point p. Let $M_p = \mathbb{C}^2 - \mathscr{A}_p$. Then there exist inclusion maps $i_p : M_p \hookrightarrow GF_n^2$. By a result of Brieskorn [3] the maps i_p induce an isomorphism

$$\bigoplus_p H_2(M_p) \cong H_2(GF_n^2).$$

Note that each relation in $\pi_1(GF_n^2)$ arises from an intersection point p of $\mathscr{A}_b$. We are thus reduced to the complement of n lines through the origin in $\mathbb{C}^2$. Denote this space by M. The dimension of $H_2(M)$ is equal to $n-1$ [15]. Now, consider the dimension of the space $\mathscr{L} \subset \Lambda^2 H_1(M)$ spanned by the linearized relations for $\pi_1(M)$. The linearizations take the form

$$a_i \wedge (a_1 + \cdots + a_n) \quad \text{for } 1 \le i \le n,$$

where a_i generates $H_1(M)$. However, the sum of all of these is equal to zero. Thus, the dimension of $\mathscr{L}$ is equal to $n-1$, as required.

Now consider the map β_1. The quotient L_1/L_0 is isomorphic to $H_1(GF_n^2) \otimes H_1(X_n^2)$. But this is clearly isomorphic to $\tilde{L}_1/\tilde{L}_0$ via β_1 by Eq. (24). The map β_2 is an isomorphism by induction. We conclude that the map φ_{n*} is also an isomorphism.

We now complete the proof by considering integer coefficients. First, since the map $H_2(PL_n) \to \Lambda^2 H_1(PL_n)$ is injective, the dimension of $H_2(PL_n)$ is equal to N. Thus, $H_2(PL_n, \mathbb{Z})$ is torsion free and the map $D: H_2(PL_n, \mathbb{Z}) \to \Lambda^2 H_1(PL_n, \mathbb{Z})$ is well defined. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} H_2(PL_n, \mathbb{Z}) & \longrightarrow & \Lambda^2 H_1(PL_n, \mathbb{Z}) \\ \Big\downarrow \varphi_{n*} & & \Big\downarrow \cong \\ H_2(X_n^2, \mathbb{Z}) & \longrightarrow & \Lambda^2 H_1(X_n^2, \mathbb{Z}). \end{array}$$

Let $I_1 = \mathrm{Im}\{D: H_2(PL_n, \mathbb{Z}) \to \Lambda^2 H_1(PL_n, \mathbb{Z})\}$, and $I_2 = \mathrm{Im}\{D: H_2(P_n^2, \mathbb{Z}) \to \Lambda^2 H_1 (X_n^2, \mathbb{Z})\}$. To complete the proof it suffices to show that $I_1 = I_2$. Note that $I_1 \subset I_2$. Thus, it suffices to show that I_1 is primitive in I_2, i.e.

$$I_1 = (I_1 \otimes \mathbb{Q}) \cap I_2.$$

This is easily seen using the isomorphism $Gr_\bullet^L I_1 \cong (Gr_\bullet^L I_1 \otimes \mathbb{Q}) \cap I_2$. □

8. Infinitesimal vector braid relations

We begin by recalling the infinitesimal presentation of the classical pure braid group. Given a group G we denote its group algebra over $\mathbb{C}$ by $\mathbb{C}[G]$. Let $\varepsilon: \mathbb{C}[G] \to \mathbb{C}$ be the augmentation homomorphism and J be equal to the kernal of ε. The powers of J define a topology on $\mathbb{C}[G]$ which is called the J-adic topology. In what follows, we let $\mathbb{C}\langle Y_i \rangle$ denote the free associative, non-commutative algebra in the indeterminants Y_i and $\mathbb{C}\langle\langle Y_i \rangle\rangle$ denote the non-commutative formal power series ring in the indeterminants Y_i. In [13] Kohno proves that the J-adic completion of the group ring $\mathbb{C}[P_n]$ is isomorphic to $\mathbb{C}\langle\langle X_{ij} \rangle\rangle$, $1 \le i < j \le n$, modulo the two-sided ideal generated by the relations

$$[X_{ij}, X_{ik} + X_{jk}] \quad \text{when } i,j,k \text{ are distinct},$$

$$[X_{ij}, X_{rs}] \quad \text{when } i,j,r,s \text{ are distinct}.$$

We now find the corresponding infinitesimal presentation for the group P_n^2.

Proposition 8.1. *The completed group ring of P_n^2 is isomorphic to $\mathbb{C}\langle\langle X_{ijk} \rangle\rangle$, $1 \le i < j < k \le n$, modulo the two-sided ideal generated by the relations*

$$[X_{ijk}, X_{rst}] \quad \textit{when } i,j,k,r,s,t \textit{ are distinct}, \tag{25}$$

$$[X_{ijk}, X_{rsk}] \quad \textit{when } i,j,k,r,s \textit{ are distinct}, \tag{26}$$

$$[X_{ijk}, X_{jkl} + X_{ikl} + X_{ijl}] \quad \textit{when } i,j,k,l \textit{ are distinct} \tag{27}$$

and

$$[X_{rst}, X_{1ij} + \cdots + X_{i-1,ij} + X_{i,i+1,j} + \cdots + X_{i,j-1,j} + X_{ij,j+1} + \cdots + X_{ijn}], \tag{28}$$

which holds when rst is one of the triples

$$\{1ij\},\dots,\{i-1,ij\},\{i,i+1,j\},\dots,\{i,j-1,j\},\{ij,j+1\},\dots,\{ijn\},$$

where $1<i<j<n$.

Proof. This is a special case of a result due to K.-T. Chen [5]. We apply the version of this result which appears in [9, pp. 28–29]. Observe that the tensor algebra $\bigoplus_{n=0}^{\infty} H_1(X_n^2,\mathbb{C})^{\otimes n}$ on $H_1(X_n^2,\mathbb{C})$ is isomorphic to $\mathbb{C}\langle X_{ijk}\rangle$, where the indeterminant X_{ijk} denotes the homology class of the generator a_{ijk} of the group P_n^2. This is a direct consequence of Lemma 7.11 and Theorem 7.14. Let

$$\delta : H_2(X_n^2,\mathbb{C}) \to H_1(X_n^2,\mathbb{C})^{\otimes 2} \subset \mathbb{C}\langle X_{ijk}\rangle$$

be the dual of the cup product. According to [9, pp. 28–29], the completed group ring of P_n^2 is isomorphic to $\mathbb{C}\langle\langle X_{ijk}\rangle\rangle/(\operatorname{im}\delta)$. Using the results contained in the proof of Theorem 7.20 the ideal $(\operatorname{im}\delta)$ is simply the two-sided ideal generated by the relations (25)–(28). □

9. Affine versus projective revisited

As we saw in Section 2, we can consider motions of points in $\mathbb{A}^m$ as being motions of points in $\mathbb{P}^m$. This has some interesting consequences for the groups P_n^2 and Q_n^2. First, note that we have the natural surjective map from $P_n^2 \to Q_n^2$ (see Lemma 2.1). Thus, we immediately see that Q_n^2 is generated by a_{ijk}, for $1 \le i<j<k \le n$. However, since we are now looking at points in $\mathbb{P}^2$ we get some extra relations amongst these generators, which are analogous to relations (4) of Q_n.

Lemma 9.1. *For* $1 \le k \le n$, *the following relations hold in* Q_n^2:

$$a_{123}\, a_{124} \dots a_{12n}\, a_{134} \dots a_{13n} a_{145} \dots a_{1,n-1,n} = 1,$$

$$a_{123}\, a_{124} \dots a_{12n}\, a_{234} \dots a_{23n} a_{245} \dots a_{2,n-1,n} = 1,$$

$$a_{12k}\, a_{13k} \dots a_{k-2,k-1,k}\, a_{1k,k+1} \dots a_{k-1,k,k+1} \dots a_{1kn} \dots a_{k-1,kn} = 1. \tag{29}$$

Proof. Each of these relations arises from a product relation which occurs in the fiber of the projection, $Y_n^2 \to Y_{n-1}^2$, obtained by forgetting the kth point. These can then be written in the required form using the reciprocity law. See Section 12.4 for more details. □

We define a group QL_n^2 by adjoining the extra relations (29) to the presentation of PL_n, and conjecture that this group is isomorphic to Q_n^2. An argument similar to that in the proof of Theorem 7.14 can be used to show that the first homology groups of QL_n and Q_n^2 are the same, and are free abelian of rank $\binom{n}{3} - n$.

We can exploit the action of $PGL_3(\mathbb{C})$ on Y_n^2 to better understand the group Q_n^2. As we have already seen, the action can be used to show that the group Q_n^2 has a cental element of order three (see Lemma 2.6). Denote this element by τ. In fact, by analysing the action of $PGL_3(\mathbb{C})$ on Y_n^2 we can find an explicit formula for this element in terms of the generators a_{ijk}.

Lemma 9.2. *The element* τ *is given by the formula*

$$\tau = \tau_3 \ldots \tau_n,$$

where

$$\tau_i = a_{12i} a_{13i} \ldots a_{i-2,i-1,i}.$$

Proof. We analyze where the map $PGL_3(\mathbb{C}) \to Y_n^2$ sends the generator $\rho : S^1 \to PGL_3(\mathbb{C})$ of $\pi_1(PGL_3(\mathbb{C}))$, given by the formula

$$\rho : \theta \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{bmatrix},$$

where $0 \leq \theta \leq 2\pi$.

Choose coordinates (x, y), $x, y \in \mathbb{R}$ for the affine part of $\mathbb{P}^2(\mathbb{R})$. Let the points b_1 and b_2 be equal to $(0,0)$ and $(0,1)$, respectively. Denote the coordinate of the point b_i by (x_i, y_i), for $i \geq 3$. The line $L_i^\infty(\mathbb{R})$ is equal to the vertical line passing through (x_i, y_i). We "stretch out" the points b_i on the curve $\psi(\mathbb{R})$ so that they satisfy the following condition. We require that the line $L_{i,i+1}(\mathbb{R})$ intersects the line $L_{i-1}^\infty(\mathbb{R})$ at a point whose y coordinate is less than $-y_{i-1}$. Note we may do this whilst remaining within the basepoint set B which was defined in Section 6.

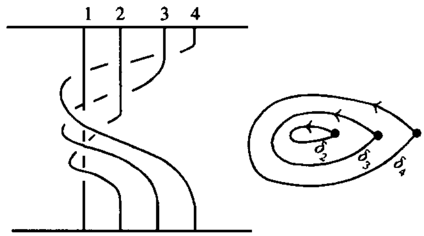
The loop ρ will be sent to the loop $\tau : S^1 \to Y_n^2$ given by the formula

$$\tau : \theta \to ((0,0), (0,1), (x_3, e^{i\theta} y_3), \ldots, (x_n, e^{i\theta} y_n)).$$

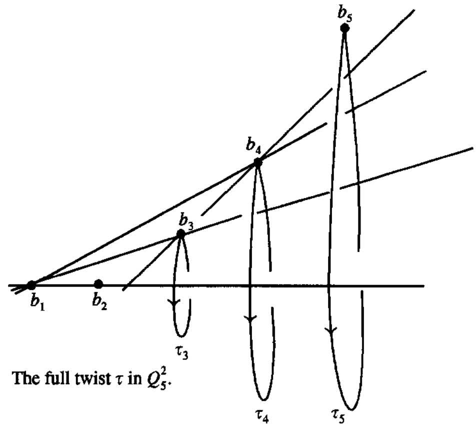
On "squeezing" the points back to their original position, we see that the loop which each point b_i follows in the loop τ is homotopic to the loop

$$\tau_i = a_{12i} a_{13i} \ldots a_{i-2,i-1,i}.$$

We now show that the loops τ_i commute with each other. Fix $i < j$. Note that the loop τ_j encircles the lines L_{rs} for $1 \leq r < s \leq j-1$. As the point b_i follows the loop τ_i the lines L_{rs} always remain within the loop τ_j. Thus, the motion of the point b_i along τ_i is independent of the loop τ_j.



The full twist Δ in Q_4.



The full twist τ in Q_5^2.

Fig. 12. Let's do the twist!.

Finally, note that since the τ_i commute we have the expression

$$\tau = \tau_3 \ldots \tau_n,$$

as required. □

The element τ of order three in Q_n^2 is analogous to the full-twist Δ of the classical braid group of the sphere. We recall some results about the classical braid groups [1]. In the classical case P_n is a group with center $\mathbb{Z}$ generated by the full-twist

$$\Delta = \delta_2 \ldots \delta_n,$$

where

$$\delta_i = a_{1i}a_{2i} \ldots a_{i-1,i}.$$

The elements δ_i commute with one other (see Fig. 12). Moreover, Δ is a central element of Q_n which has order 2.

Using the classical braid groups as a model, it is natural to conjecture that τ generates the center of the group P_n^2.

10. Conjugation

In this section we describe a method for conjugating generators of $\pi_1(GF_n^2)$ by generators of the group P_{n-1}^2.

It will be helpful to first discuss the classical pure braid case. Recall the short exact sequence (2). To find a presentation for P_n from one for P_{n-1}, we have to be able to write $a_{ij}^{-1}a_{rn}a_{ij}$ as a word in the group L_{n-1}, where $1 \le i < j \le n-1$, and $1 \le r \le n-1$. We can do this by picturing the loops a_{rn} and a_{ij} in $\mathbb{C}$ (Fig. 13). As we "unwind" the braid $a_{ij}^{-1}a_{rn}a_{ij}$, the point j moves around point i and "pushes" the loop a_{rn} with it. At the end of the unwinding process, we are left with a loop in L_{n-1}, which will be the required conjugate of a_{rn}. We can describe this loop in terms of the fiber generators by recording where this loop crosses vertical half-lines below the points $\{1,\dots,n-1\}$. For example, way in which the relation

$$a_{ij}^{-1}\, a_{rs}\, a_{ij} = a_{is}\, a_{rs}\, a_{is}^{-1}, \quad 1 \le i < r = j < s \le n,$$

in the group P_n is obtained is shown in Fig. 14.

We now describe a similar process for conjugating in P_n^2. Let $L(\mathbb{R})$ denote the line $L_n^\infty(\mathbb{R})$ which we used to define the loop a_{rsn}, $1 \le r < s \le n-1$. Let p_{rs} denote the intersection of line $L_{rs}(\mathbb{R})$ with $L(\mathbb{R})$. We picture the generator a_{rsn} contained in L in Fig. 15.

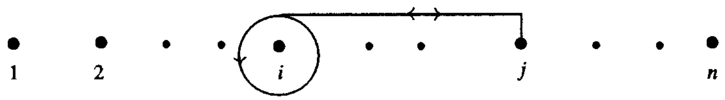


Fig. 13. The generator a_{ij} of P_n.

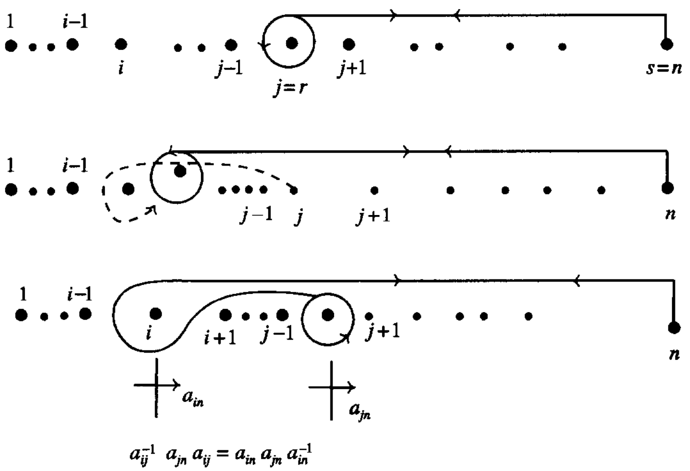


Fig. 14. Conjugation in P_n.

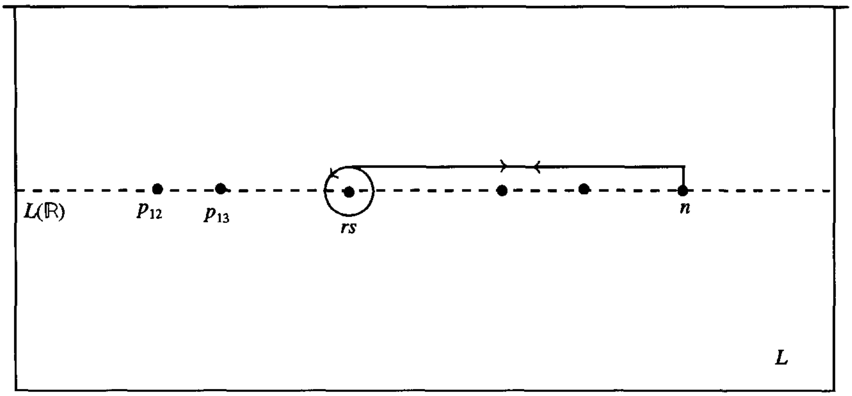


Fig. 15. The generator a_{rsn} of P_n^2.

Note that if we move the point b_k for $1 \leq k \leq n-1$, then we will induce a motion of the points p_{rs} within L. Hence, when point b_k follows the loop a_{ijk}, we can picture the induced motions of the points p_{rs} in L. We can use this picture to determine how generator a_{ijk} conjugates generator a_{rsn} by recording how the movements of the p_{rs} within L deform the loop a_{rsn}.

At this point we need to make some observations which will simplify the calculation. First, note that the motion of the point b_k only induces a motion of the points p_{rk}. Moreover, by choosing the loop l which the point b_k follows on a_{ijk} whilst going around $L_{ij}(\mathbb{R})$ to be small enough (recall the definition of the generator a_{ijk}), we can ensure that only the points p_{jk} and p_{ik} go around p_{ij} as the point b_k goes around l. Hence, we need only consider the motion of the points p_{ik} and p_{jk} within L. We picture the motion of these points induced by the motion of b_k in Fig. 16.

In Fig. 17 we picture how to obtain the relation

$$a_{ijk}^{-1} a_{rsn} a_{ijk} = a_{ijn} a_{ikn} a_{rsn} a_{ikn}^{-1} a_{ijn}^{-1}, \quad jk = rs$$

in the group P_n^2. The other conjugation relations are found in the a similar way.

11. The reciprocity law

In this section we define a homotopy between two loops within X_n^2, which we call the *reciprocity law*. This will be used in Section 12 to lift certain relations from P_{n-1}^2 to P_n^2.

To understand the reciprocity law, it is helpful to understand the analogous law within the classical pure braid group. We may consider the generator $a_{ij} \in P_n$ to be the braid whose jth string passes around the ith string. However, by "pulling the jth string tight" we can also consider this braid to be the one whose ith string passes around

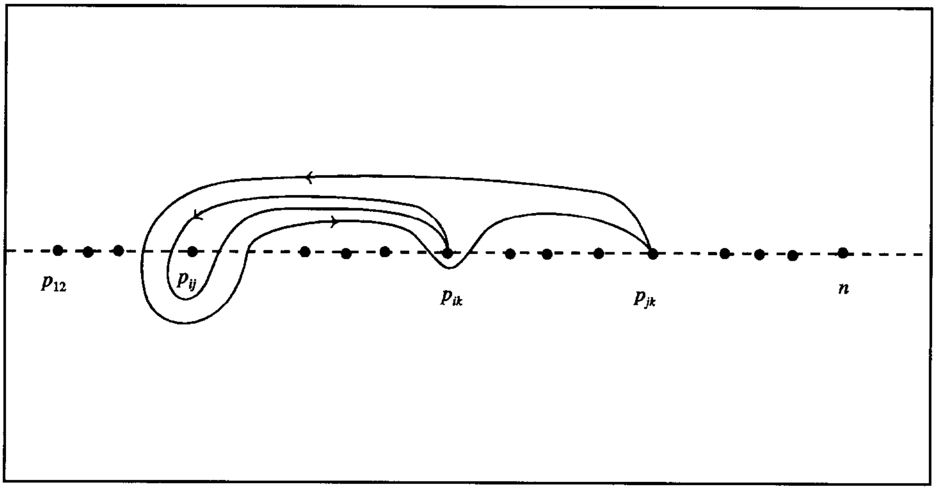
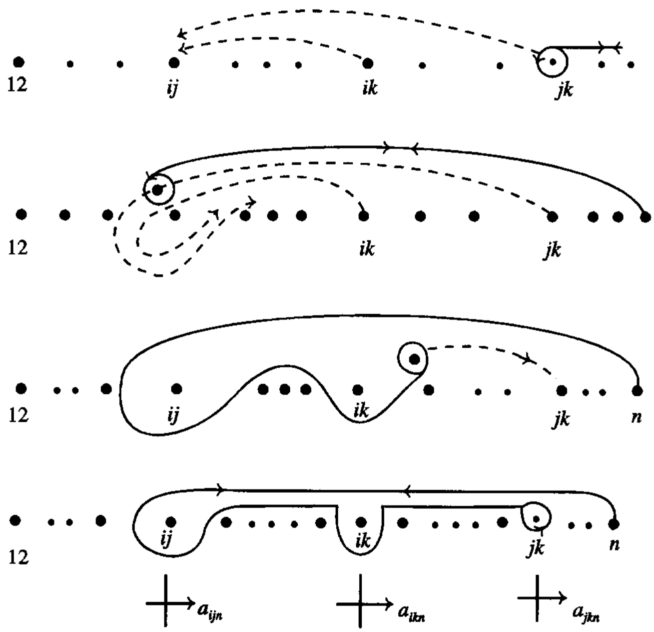


Fig. 16. Induced motion.



$$a_{ijk}^{-1} \quad a_{jkn} \, a_{ijk} = a_{ijn} \, a_{ikn} \, a_{jkn} \, a_{ikn}^{-1} \quad a_{ijn}^{-1}$$

Fig. 17. Conjugation in P_n^2.

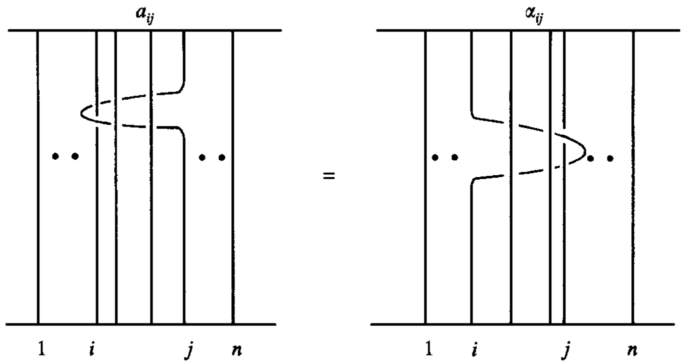


Fig. 18. The reciprocity law for P_n.

the jth string (see Fig. 18). We label this new loop α_{ij}. The reciprocity law for P_n is simply the statement that the loop a_{ij} is homotopic to the loop α_{ij} in X_n^1.

Now, we describe the reciprocity law for P_n^2. First, we need to define a new loop α_{ijk} in X_n^2. Let F_j be the fiber defined in Section 6. Choose a point $q \in I(\mathbb{R})$ on the line $L_{ik}(\mathbb{R})$ within the disc of radius ε about the point b_k. Let the line $L'(\mathbb{R})$ denote the real line joining b_j and q. Then we define the loop α_{ijk} to be the loop in L', based at b_j, which goes around q. To prove the reciprocity law for P_n^2 we will use the following key lemma.

Lemma 11.1. *The two loops $a_{i,k-1,k}$ and $\alpha_{i,k-1,k}$ are homotopic in X_n^2 relative to the basepoint b, for $2 \leq i < k \leq n$ (see Fig. 19).*

The proof is computational, and is given in Section 13.

Proposition 11.2 (The reciprocity law). *The loops a_{ijk} and α_{ijk} are homotopic in X_n^2 relative to the basepoint $b \in B$, for $1 \leq i < j < k \leq n$ (see Fig. 20).*

Proof. We find an explicit homotopy. Commence the homotopy by shrinking the loop a_{ijk}. Let $\varepsilon > 0$ be a small real number. Let $L(\mathbb{R})$ denote the line L_k^{∞} which was used to define the loop a_{ijk}. Let p be a point on $\psi(\mathbb{R})$ lying between the points b_{k-1} and b_k. Let $\tilde{L}(\mathbb{R})$ be the line joining the points b_j and p. Let v be a vector which orients the line $\tilde{L}(\mathbb{R})$ in the direction from b_j to p. Move the point b_j up distance ε into $\tilde{L}$ in the direction iv. Then move b_j along the line above $\tilde{L}(\mathbb{R})$ until it reaches the point in $\tilde{L}$ which is distance ε in the direction iv above p. Finally, move b_j back down to p. During the motion of b_j the points $p_{ij} = L \cap L_{ij}$ will move within the line L. Shrink the loop a_{ijk}, so that it follows the motions of the p_{ij}.

By sliding points along $\psi(\mathbb{R})$ whilst remaining in the base point set B, if necessary, we will now be in the same situation as Lemma 11.1 with $j = k$. The shrunken version of a_{ijk} is thus homotopic to the loop $\alpha_{i,k-1,k}$. By sliding points again, if necessary, we now move the point b_j back to its starting position along the same path which

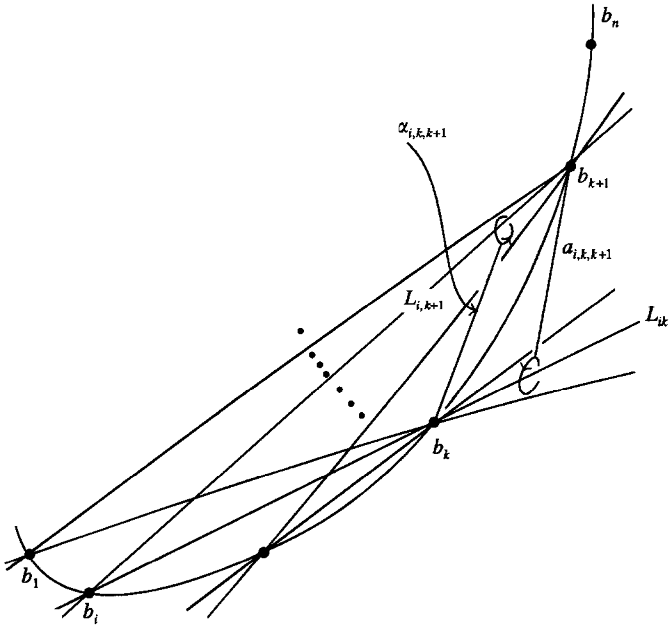


Fig. 19. Reciprocity.

it originally took. During this process the loop $\alpha_{i,k-1,k}$ will get "stretched out". The resulting loop is homotopic to α_{ijk}. □

12. Lifting relations

In this section we show how to lift relations from the group P_{n-1}^2 to P_n^2 in order to complete the proof of Theorem 7.2. Note that unlike sequence (2) for the pure braid group, which is split, up until now it has not been possible to find a splitting for the sequence (3). Hence, it is necessary to overcome the problem of lifting relations from P_{n-1}^2 to P_n^2 in an alterative way. We also show how to lift relation (29) from Q_{n-1}^2 to Q_n^2. Before we begin lifting relations, we describe the main idea that we shall use.

First, consider the classical pure braid groups. We may lift a relation from P_{n-1} to P_n, if the nth string does not obstruct the homotopy describing this relation in P_{n-1}. Since the nth string does not obstruct any of the homotopies describing the relations in P_{n-1}, they all lift.

A similar idea applies in lifting relations from P_{n-1}^2 to P_n^2, although in this case we get some obstructions. The point b_n introduces the lines $L_{in}(\mathbb{C})$ into GF_n^2. If it is possible to describe a relation in P_{n-1}^2 by a homotopy within F_k, $1 \leq k \leq n-1$, which does not intersect any of the L_{in}, then we can lift this relation. This is because we

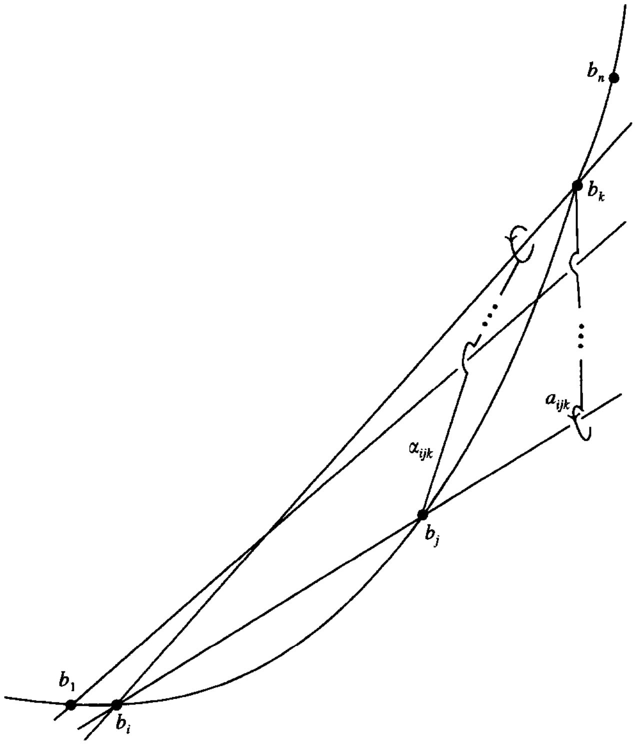


Fig. 20. The reciprocity law for P_n^2.

have the sequence[4]

$$\pi_1(F_k, b_k) \to P_n^2 \to P_{n-1}^2 \to 1.$$

However, if it is impossible to describe a relation in P_{n-1}^2 by a homotopy which does not intersect the L_{in}, then we have to use the reciprocity law to lift the relation.

12.1. Relations (15)–(17)

Lemma 12.1. *For* $1 \le k \le n-1$ *the relations* (15)–(17) *lift from* P_{n-1}^2 *to* P_n^2.

Proof. For $1 \le k \le n-1$ the relations (15)–(17) hold in $\pi_1(F_k)$. □

12.2. Relation (19)

Lemma 12.2. *For* $1 \le k \le n-1$ *the relation* (19) *lifts from* P_{n-1}^2 *to* P_n^2.

[4] See Section 6 for the definition of F_k.

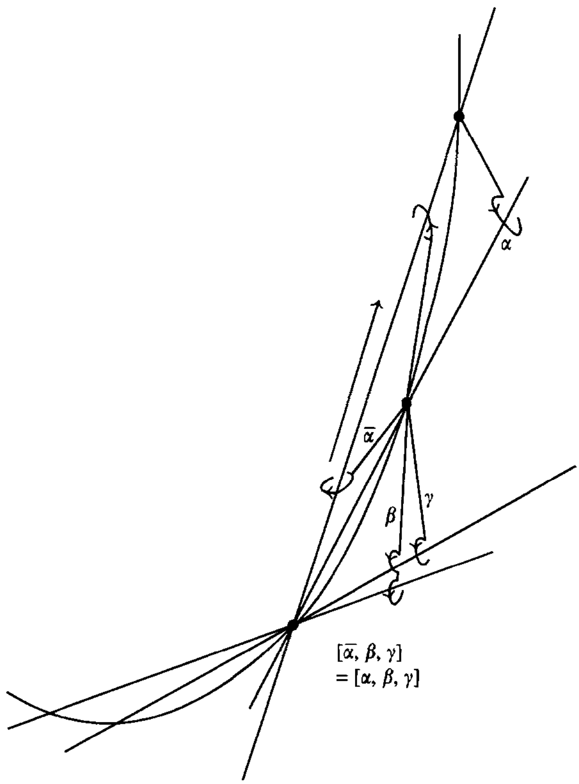


Fig. 21. How Hopf relations lift.

Proof. Relation (19) is described by a homotopy $H : [0,1] \times [0,1] \to X^2_{n-1}$. Let $R_l \subset L^\infty_l$ be equal to $(\operatorname{im} H) \cap L^\infty_l$. Then the homotopy H can be chosen so that the lines L_{in} within F_k do not intersect the region R_l for $l = j,k$. This is because the points $b_1, \dots, b_n$ satisfy the lexcigon condition. Hence, H lifts to X^2_n. □

12.3. Relation (18)

This case is not the same as the previous two, since the lines L_{in} in F_k obstruct the homotopy describing relation (18) within P^2_{n-1}. This is because the lines L_{in} always pass through the point b_i. First note that, for fixed k and j, the relation

$$[a_{ijk}, \dots, a_{j-1,jk}, \tilde{a}_{j,j+1,k}, \dots, \tilde{a}_{j,k-1,k}, \alpha_{jk,k+1}, \dots, \alpha_{jk,n-1}] = 1 \tag{30}$$

holds in $\pi_1(F_k)$ (and so also in P^2_n). This relation arises from the Hopf link of the point b_j. To understand why this is the case see Fig. 21. Now, using the reciprocity law relation (30) can be rewritten as follows:

$$[a_{ijk}, \dots, a_{j-1,jk}, \tilde{a}_{j,j+1,k}, \dots, \tilde{a}_{j,k-1,k}, a_{jk,k+1}, \dots, a_{jk,n-1}] = 1.$$

Thus, relation (18) holds P_n^2. Now we simply note that

$$(p_n^2)_*([a_{ijk},\dots,a_{j-1,jk},\tilde{a}_{j,j+1,k},\dots,\tilde{a}_{j,k-1,k},a_{jk,k+1},\dots,a_{jk,n-1}])$$
$$= [a_{ijk},\dots,a_{j-1,jk},\tilde{a}_{j,j+1,k},\dots,\tilde{a}_{j,k-1,k},a_{jk,k+1},\dots,a_{jk,n-2}]$$

for $1 \le k \le n-1$. Hence, we have lifted relation (18).

12.4. Relation (29)

We conclude this section by showing how to lift the product relations (29) from Q_{n-1}^2 to Q_n^2. We can use the same method that we used to lift relation (18). Let G_k denote the fiber of the projection, $Y_n^2 \to Y_{n-1}^2$, obtained by forgetting the kth point. Then for each k the relation (29) holds in $\pi_1(G_k)$ as a consequence of the reciprocity law. Thus relation (29) holds in Q_n^2. Now note that,

$$(p_n^2)_*(a_{12k}\, a_{13k} \cdots a_{k-2,k-1,k}a_{1k,k+1} \cdots a_{k-1,k,k+1} \cdots a_{1kn} \dots a_{k-1,kn})$$
$$= a_{12k}\, a_{13k} \cdots a_{k-2,k-1,k}a_{1k,k+1} \cdots a_{k-1,k,k+1} \cdots a_{1k,n-1} \cdots a_{k-1,k,n-1}$$

for $1 \le k \le n-1$. Hence, we have lifted relation (29).

It is informative to understand how this relation lifts. First, let us understand the analogous situation in the pure braid group on $\mathbb{P}^1$. In Fig. 22 we picture how to lift the relation

$$a_{1,n-1}a_{2,n-1} \cdots a_{n-2,n-1} = 1, \tag{31}$$

which holds in Q_{n-1}, to the relation

$$a_{1n}a_{2n} \cdots a_{n-1,n} = 1,$$

which holds in Q_n. Note that the nth string will always obstruct any homotopy which describes relation (31) in the group Q_{n-1}.

We now show how to lift relation (29). First, consider this relation in Q_{n-1}^2. This may be rewritten as the product

$$a_{12k} \cdots a_{1,k,n-1}a_{23k} \cdots a_{k-2,k-1,k}\alpha_{k,k+1,k+2} \cdots \alpha_{k,n-1,n-2} \tag{32}$$

using the reciprocity law. We see that a homotopy describing relation (29) can be chosen to lie in the line L_k^∞ (Fig. 23). However, the lines L_{in} intersect $L_k^\infty(\mathbb{R})$, and so the product (32) will be homotopic to a loop which encircles the points $L(\mathbb{R}) \cap L_{in}(\mathbb{R})$, within L_k^∞. This is homotopic to the product

$$\alpha_{k,n-1,n}^{-1} \dots \alpha_{k,k+1,n}^{-1}\alpha_{k-1,k,n}^{-1} \dots \alpha_{1,k,n}^{-1}.$$

The reciprocity law allows us to rewrite this expression in terms of the a_{ijn}. Thus, we have lifted relation (29).

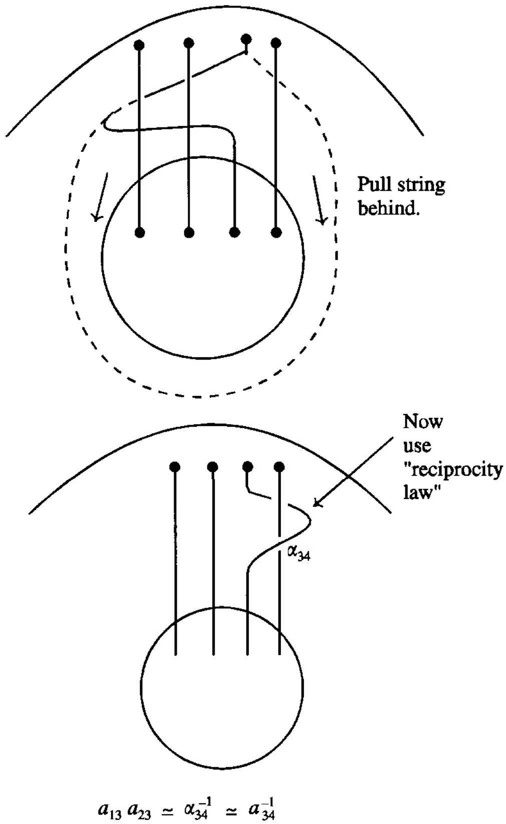


Fig. 22. Lifting a product from Q_{n-1} to Q_n.

13. Proof of Lemma 11.1

We prove Lemma 11.1 in the following way. In Section 13.1, we choose a base point x in the set $B \subset X_n^2$. We then define two loops ξ_m, $\tilde{\xi}_m$, based at x, and show that these loops are homotopic relative to x. In Section 13.2, we generalize the result of Section 13.1. Finally, in Section 13.3, we use the results of Sections 13.1 and 13.2 to prove Lemma 11.1.

13.1. We begin by choosing the base point x in $B \subset X_n^2$. Note that $X_n^2 \subset (\mathbb{C}^2)^m$. The curve ψ is then given by the equation $\psi(t) = t^2$. Let

$$(x_m, y_m) = (-m/2n, \psi(-m/2n)) = (-m/2n, (m/2n)^2), \quad 1 \le m \le n-2.$$

Choose $\varepsilon \in \mathbb{R}$ to be a sufficiently small positive number. Let $u = ((1-\varepsilon), (1-\varepsilon)^2)$ and $v = ((1+\varepsilon), (1+\varepsilon)^2)$. Then, we define $x \in X_n^2$ to be the point

$$((x_1, y_1), \ldots, (x_{n-2}, y_{n-2}), u, v).$$

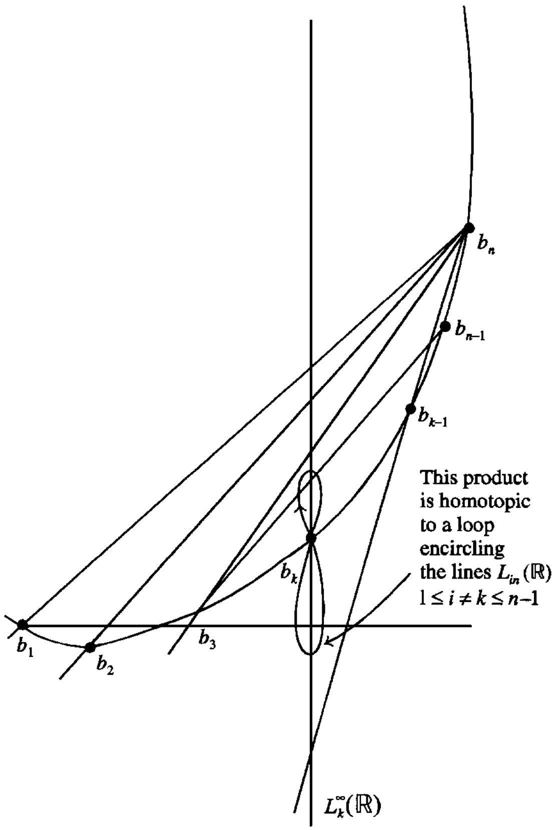


Fig. 23. Lifting a product from Q_{n-1}^2 to Q_n.

We now define two loops ξ_m and $\tilde{\xi}_m$ in X_n^2, and show that they are homotopic relative to x.

Let p_m be the y co-ordinate of the intersection of the line $x=(1+\varepsilon)$ and the line joining points u and (x_m, y_m). Then we have

$$p_m = 2\varepsilon((1-\varepsilon)-(m/n))+(1-\varepsilon)^2.$$

Let q_m be the y co-ordinate of the intersection of the line $x=(1-\varepsilon)$ and the line joining points v and (x_m, y_m). Then we have

$$q_m = -2\varepsilon((1+\varepsilon)-(m/n))+(1+\varepsilon)^2.$$

Let

$$r_m = (1+\varepsilon)^2 - (p_m + p_{m+1})/2$$

and

$$s_m = (q_m + q_{m+1})/2 - (1-\varepsilon)^2.$$

These will be the "radii" of the loops ξ_m and $\tilde{\xi}_m$, respectively (see Fig. 24).

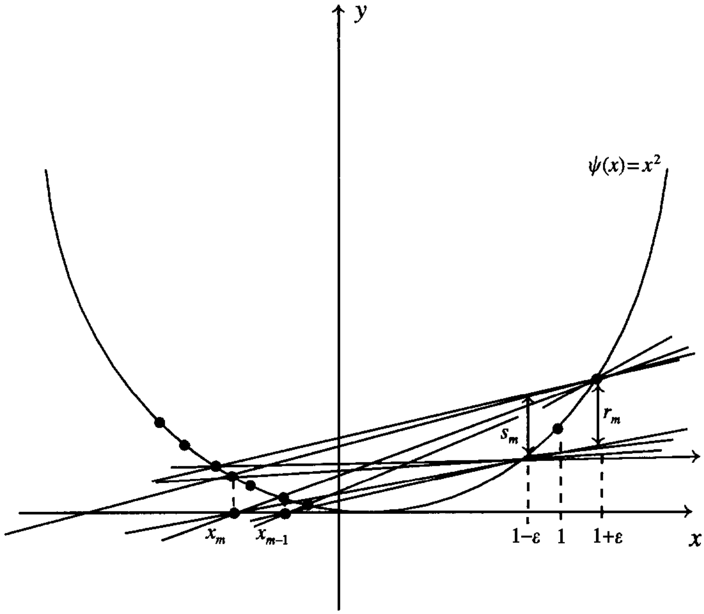


Fig. 24.

Define

$$\xi_m(t) = ((1+\varepsilon), (1+\varepsilon)^2 + r_m(e^{it}-1)/2) \quad \text{for } 0 \le t \le 2\pi$$

and

$$\tilde{\xi}_m(t) = ((1-\varepsilon), (1-\varepsilon)^2 + s_m(1-e^{it})/2) \quad \text{for } 0 \le t \le 2\pi.$$

A homotopy, $H_m(s,t)$, where $0 \le s \le 1$ and $0 \le t \le 2\pi$, between the loops ξ_m and $\tilde{\xi}_m$ is now given explicitly by

$$H_m(s,t) = ((x_1,y_1),\ldots,(x_{n-2},y_{n-2}),(u_1(s,t),u_2(s,t)),(v_1(s,t),v_2(s,t))),$$

where

$$(u_1(s,t),u_2(s,t)) = ((1-\varepsilon),(1-\varepsilon)^2 + (1-s)s_m(1-e^{it})/2)$$

and

$$(v_1(s,t),v_2(s,t)) = ((1+\varepsilon),(1+\varepsilon)^2 + sr_m(e^{it}-1)/2).$$

Note that $H_m(0,t) = \xi_m(t)$ and $H_m(1,t) = \tilde{\xi}_m(t)$ for $0 \le t \le 2\pi$. Also, note that $H_m(s,0) = x$ and $H_m(s,2\pi) = x$ for $0 \le s \le 1$. We are thus reduced to showing that the homotopy lies in X_n^2.

Let $(x_1,y_1),\ldots,(x_n,y_n)$ be n points in $\mathbb{C}^2$. We will need to check whether three of these points lie on a line. The following condition will be convenient for our calculations. No three of these points will lie on a line if and only if no 3×3 minor of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

vanishes. Hence, to prove that the loops ξ_m and $\tilde{\xi}_m$ are homotopic, it will be sufficient to show that no 3×3 minor the matrix

$$H(s,t) = \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 \\ x_1 & \cdots & x_{n-2} & u_1(s,t) & v_1(s,t) \\ y_1 & \cdots & y_{n-2} & u_2(s,t) & v_2(s,t) \end{bmatrix}$$

vanishes for any $0 \le t \le 2\pi$ and $0 \le s \le 1$.

We split this calculation up into three cases. Let C_i denote the ith column of H, for $1 \le i \le n$. Since the points corresponding to columns $C_1, \ldots, C_{n-2}$ lie on $\psi(\mathbb{R})$ and are fixed for all s and t, it suffices to check that the determinants of the following matrices do not vanish, for all $1 \le k < l \le n-2$:

$$[C_k, C_l, C_{n-1}], \tag{33}$$

$$[C_k, C_l, C_n], \tag{34}$$

$$[C_k, C_{n-1}, C_n]. \tag{35}$$

13.1.1. Determinant of (33)

It suffices to check this only when $t = \pi$, since otherwise the determinant of matrix (34) has non-zero complex part and is thus non-zero. Let t be equal to π. Then matrix (33) is equal to

$$\begin{bmatrix} 1 & 1 & 1 \\ -(k/2n) & -(l/2n) & (1-\varepsilon) \\ (k/2n)^2 & (l/2n)^2 & (1-\varepsilon)^2 + (1-s)s_m \end{bmatrix}.$$

Thus, we need only show that the determinant of this matrix does not vanish for any $0 \le s \le 1$. Expanding out the determinant of this matrix gives us the following expression:

$$(k/2n - l/2n)((kl/4n^2) + (1-\varepsilon)^2 + (1-\varepsilon)(l/2n + k/2n) - (1-s)s_m).$$

Suppose that this expression is equal to zero for some s. Then, by choosing ε sufficiently small we can force the inequality

$$s > (1 + 1/2n)^2.$$

But $0 \le s \le 1$, and we are lead to a contradiction.

13.1.2. Determinant of (34)

It suffices to check this only when $t = \pi$, since otherwise the determinant of matrix (34) has non-zero complex part and is thus non-zero. Let t be equal to π. Then matrix (34) is equal to

$$\begin{bmatrix} 1 & 1 & 1 \\ -(k/2n) & -(l/2n) & (1+\varepsilon) \\ (k/2n)^2 & (l/2n)^2 & (1+\varepsilon)^2 + sr_m \end{bmatrix}.$$

Thus, we need only show that the determinant of this matrix does not vanish for any $0 \leq s \leq 0$. Expanding out the determinant of this matrix gives us the following expression:

$$(k/2n - l/2n)((kl/4n^2) + (1+\varepsilon)^2 + (1+\varepsilon)(l/2n + k/2n) - sr_m).$$

Suppose that this expression is equal to zero for some s. Then, by choosing ε sufficiently small, we can force the inequality

$$s > (1 + 1/2n)^2.$$

But $0 \leq s \leq 1$, and we are lead to a contradiction.

13.1.3. Determinant of (35)

This case is more complicated since the imaginary part of the determinant of matrix (35) may vanish. By simplifying the determinant, we obtain the following expression for the imaginary part of the determinant of matrix (35):

$$-\sin t((s_m(1 + l/2n + \varepsilon) + s((1 + l/2n - \varepsilon)r_m - (1 + l/2n + \varepsilon)s_m))/2). \tag{36}$$

Note that $\sin t$ can only vanish when $t = 0$ or when $t = \pi$. We explain what happens in the case when $t = \pi$ later. Suppose that t is not equal to 0 or π and that expression (36) vanishes. In this case s would be equal to

$$-s_m(1 + l/2n + \varepsilon)/((1 + l/2n - \varepsilon)r_m - (1 + l/2n + \varepsilon)s_m).$$

A computation shows that this quantity must be greater than one. Hence, we are lead to a contradiction.

The final case we are left with is when $t = \pi$. In this case, by supposing that the determinant vanishes, we obtain

$$sK = 1 - 2\varepsilon(1 + l/n - \varepsilon)/s_m, \tag{37}$$

where,

$$K = (((1 + l/2n - \varepsilon)r_m)/((1 + l/2n + \varepsilon)s_m) - 1).$$

A computation shows that K cannot be equal to zero, and hence we may divide Eq. (37) by K. Again, we are left with s equal to a quantity which, by allowing ε to be sufficiently small, may be shown to be greater than one. Thus, the image of the map H_k lies in X_n^2 and the loops ξ_m and $\tilde{\xi}_m$ are homotopic.

13.2. We now generalize the result of Section 13.1. To prove Lemma 11.1, we will need to place points on the curve $\psi(\mathbb{R})$ to the right of the points u and v, without obstructing the homotopy between the loops ξ_m and $\tilde{\xi}_m$. We do this by placing our extra points on $\psi(\mathbb{R})$ so that they are "far away" from u and v. In this way, we can ensure that the lines introduced by adding in the new points do not intersect the complexified lines $x = 1 + \varepsilon$ and $1 - \varepsilon$ within the circles of radius r_m and s_m respectively. Now, it is clear that the lines introduced by adding the new points do not obstruct the homotopy between the loops ξ_m and $\tilde{\xi}_m$.

13.3. We now complete the proof of Lemma 11.1. Recall that we are trying to show that the loops $a_{i,k-1,k}$ and $\alpha_{i,k-1,k}$ are homotopic relative to b. We can slide points up and down $\psi(\mathbb{R})$ as long as we remain in the contractible set B defined in Section 6. Slide the points $\{b_{k+1}, \dots b_n\}$ along the curve $\psi(\mathbb{R})$ until they are "far to the right" of the points b_{k-1} and b_k. Now slide points b_{k-1} and b_k into positions u and v of Section 13.1. Finally, slide points $\{b_1, \dots, b_{k-2}\}$ so that we are in the situation of Section 13.2. During the sliding process the loops $a_{i,k-1,k}$ and $\alpha_{i,k-1,k}$ will be deformed; we denote the resulting loops by the same symbol.

The loop ξ_m lies in the complexification of the line $x = 1 + \varepsilon$. The loop $a_{i,k-1,k}$ can also be homotoped into this line. Similarly, the loop $\alpha_{i,k-1,k}$ can be homotoped into the complexification of the line $x = 1 - \varepsilon$, in which the loop $\tilde{\xi}_m$ lies. Hence, we are reduced to showing that certain loops are homotopic in punctured complex lines. In this situation we have

$$a_{i,k-1,k} = \xi_i \xi_{i-1}^{-1}$$

and

$$\alpha_{i,k-1,k} = \tilde{\xi}_i \tilde{\xi}_{i-1}^{-1}.$$

We know from Section 13.2 that the loops ξ_m and $\tilde{\xi}_m$ are homotopic for $1 \le m \le k-2$. Thus, the loop $a_{i,k-1,k}$ is homotopic to the loop $\alpha_{i,k-1,k}$. By sliding all points back to the basepoint b, whilst remaining in the set B, we complete the proof of Lemma 11.1.

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